Applied Mathematics and Computation 217 (2011) 7579–7587

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Weighted pseudo almost automorphic mild solutions to semilinear fractional differential equations

Gisèle M. Mophou

Laboratoire C.E.R.E.G.M.I.A., Université des Antilles et de la Guyane, Campus de Fouillole, 97159 Pointe-à-Pitre, Guadeloupe (FWI), France

ARTICLE INFO

Keywords:
Almost automorphic
Pseudo almost automorphic
Composition
Weighted pseudo almost automorphic
Abstract differential equation
Mild solution

ABSTRACT

This paper is concerned with the existence and uniqueness of weighted pseudo almost automorphic mild solution to the semilinear fractional equation: $D_t^\alpha u(t) = Au(t) + D_t^{\alpha-1} f(t, u(t), Bu(t)),\ t \in \mathbb{R},\ 1 < \alpha < 2$ where A is a linear densely defined operator of sectorial type on a complex Banach space $\mathbb{X}$ and B is a bounded linear operator defined on $\mathbb{X}$. Under the assumption of uniformly continuity on f, we establish a composition of weighted pseudo almost automorphic in a general Banach space and obtain existence results by means of Banach contraction mapping. The results obtained are utilized to study the existence and uniqueness of a weighted pseudo almost automorphic solution to fractional diffusion wave equation with Dirichlet conditions.

1. Introduction

In recent years, fractional equations have gained considerable importance due to their applications in various fields of the science, such as physics, mechanics, chemistry engineering, etc. Significant development has been made in ordinary and partial differential equations involving fractional derivatives, we refer to the monographs of Kilbas et al. [1,2], Diethelm [3], Hilfer [4], Podlubny [5] and the papers of Agarwal et al. [6,7], Benchohra et al. [8,9], El Borai [10], Lakshmikantham et al. [11–14], Mophou et al. [15–18], N'Guérékata [19] and the reference therein. Among those partial differential equations, fractional diffusion equations have attracted increasing attention. A strong motivation for studying and investigating the solution and the properties for fractional diffusion equations comes from the fact that they describe efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [20,21] and references therein), fractional random walk, etc. In [22], Oldham and Spanier discuss the relation between a regular diffusion equation and a fractional diffusion equation that contains a first order derivative in space and half order derivative in time. Mainardi [23] and Mainardi et al. [24,25] generalized the diffusion equation by replacing the first time derivative with a fractional derivative of order α. These authors proved that the process changes from slow diffusion to classical diffusion, then to diffusion-wave and finally to classical wave when α increases from 0 to 2. The fundamental solutions of the Cauchy problems associated to these generalized diffusion equation. $(0 < \alpha \leqslant 2)$ are studied in [25–27]. By means of Fourier–Laplace transforms, the authors expressed these solutions in term of Wright-type functions that can be interpreted as spatial probability density functions evolving in time with similarity properties. Agrawal [28] studied the solutions for a fractional diffusion wave equation defined in a bounded set when the fractional time derivative is described in the Caputo sense. Using Laplace transform and finite sine transform technique, the author obtained the general solution in terms of Mittag–Leffler functions. We also refer to [29,30] where fractional diffusion equations are studied by means of Mellin transform and Fourier–Laplace transform technique respectively. The study of almost automorphic solutions to fractional differential equation were initiated by Araya and Lizama [31]. In their work, the authors investigated the existence and uniqueness of an almost automorphic mild solution of the semilinear equation

E-mail address: gmophou@univ-ag.fr

0096-3003/$ - see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2011.02.048

$$D_t^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2,$$

when A is a generator of an α-resolvent family and D_t a Riemann Liouville fractional derivative. In [32] Cuevas and Lizama considered the following fractional differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2; \tag{1}$$

where A is a linear operator of sectorial negative type on a complex Banach space. Under suitable conditions on f, the authors proved the existence and uniqueness of an almost automorphic mild solution to (1). Mophou et al. [33] prove the existence and uniqueness of pseudo almost automorphic mild solution to autonomous evolution equation

$$D_t^{\alpha}(u(t) - F_1(t, B_1u(t))) = A(u(t) - F_1(t, B_1u(t)) + D_t^{\alpha-1}F_2(t, u(t), B_2u(t)), \quad t \in \mathbb{R},$$

where $1 < \alpha < 2$, A is a linear operator of sectorial negative type, and B_1, B_2 are bounded linear operators. Cuevas et al. [34,37] study respectively the pseudo almost periodic and pseudo almost periodic of class infinity mild solution to (1) assuming that $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ is a pseudo-almost periodic and pseudo almost periodic of class infinity function satisfying suitable conditions in $x \in \mathbb{X}$. See also [35,36] where the S-asymptotically ω-periodic solution to (1) is studied. Recently, Agarwal et al. [39] study the existence and uniqueness of a weighted pseudo-almost periodic mild solution to the semilinear fractional equation

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in \mathbb{R}, \quad 1 < \alpha < 2,$$

where A is a linear operator of sectorial negative type. Let us recall that the concept of weighted pseudo almost periodic functions introduced by Diagana [40] and was generalized by N'Guérékata et al. [41] to the concept of weighted pseudo almost automorphic functions. By constructing counterexamples Liang et al. [42] showed that the decomposition of such functions is not unique in general. Actually, they proved that the decomposition of weighted pseudo almost periodic functions as well as weighted pseudo almost automorphic functions is unique if the space of the ergodic components is translation invariant.

Motivated by above works, we study in this paper the existence and uniqueness of weighted pseudo almost automorphic mild solutions to the following fractional differential equation:

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t), Bu(t)), \quad t \in \mathbb{R}, \tag{2}$$

where $1 < \alpha < 2$, $A : D(A) \subset \mathbb{X} \to \mathbb{X}$ is a linear densely defined operator of sectorial type on a complex Banach space $(\mathbb{X}, \|\cdot\|)$, B is a bounded linear operator and $f : \mathbb{R} \times \mathbb{X} \times \mathbb{X} \mapsto \mathbb{X}$ is a weighted pseudo almost automorphic function in t for each $x, y \in \mathbb{X}$ satisfying suitable conditions. The fractional derivative D_t^{α} is to be understood in Riemann–Liouville sense. We first establish a composition of weighted pseudo almost automorphic functions in a general Banach space, assuming that the function f is uniformly continuous in any bounded of $\mathbb{X} \times \mathbb{X}$. Then assuming that f satisfies a Lipchitz type condition in x, $y \in \mathbb{X}$ we obtain existence results by means of the Banach contraction mapping.

The work is organized as follows. In Section 2, we recall some definitions and facts on sectorial operators, on almost, pseudo-almost and weighted pseudo almost automorphic functions. We also present some preliminary results. In Section 3 we present the main results. An example is given in Section 4 to illustrate the results obtained.

2. Preliminaries

Let $(\mathbb{X}, \|\cdot\|)$ and $(\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two complex Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$, (respectively $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $f : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm defined by

$$\|f\|_{\infty} = \sup_{t \in \mathbb{R}} \|f(t)\|$$

is a Banach space. Let also $L(\mathbb{X})$ be the Banach space of all bounded linear operators from $\mathbb{X}$ into itself endowed with the norm:

$$\|T\|_{L(\mathbb{X})} = \sup\{\|Tx\| : x \in \mathbb{X}, \|x\| \leqslant 1\}.$$

Now we give some necessary definitions.

Definition 2.1 [43,45].

1. Let $f : \mathbb{R} \mapsto \mathbb{X}$ be a bounded continuous function. We say that f is almost automorphic if for every sequence of real numbers $\{s_n\}_{n=1}^{\infty}$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that:

$$g(t) = \lim_{n \to \infty} f(t + \tau_n)$$

is well-defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} g(t - \tau_n) = f(t)$$

for each $t \in \mathbb{R}$. Denote by $AA(\mathbb{X})$ the set of all such functions.