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Application of optimal homotopy asymptotic method for the analytic solution of singular Lane–Emden type equation

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ABSTRACT

In this study, optimal homotopy asymptotic method is applied on singular initial value Lane–Emden type problems to check the effectiveness and performance of the method. It is observed that the method is easy to implement, quite valuable to handle singular phenomena and yield excellent results at minimum computational cost. Computational results of some of the test problems are presented to demonstrate the viability and practical usefulness of the method.

1. Introduction

Scientific literature embraces different techniques to model and formulate physical structures. The singular phenomenon, appears during ascertaining and modeling of certain physical structures, is of considerable importance in mathematical physics and other branches of sciences. Lane–Emden type equations, first published by Jonathan Homer Lane in 1870 [1], and further explored in detail by Emden [2], represents such phenomena and having significant applications, is a second-order ordinary differential equation with an arbitrary index, known as the polytropic index, involved in one of its terms. General form of Lane–Emden type of equations is:

$$u'' + \frac{m}{x}u' + f(u) = g(x), \quad 0 < x < 1, \quad m \geqslant 1, \tag{1}$$

subject to following initial conditions

$$u(0) = \alpha, \quad u'(0) = \beta, \tag{2}$$

where α, β and m are constants and $f(u)$ is a real valued continuous function. For $m = 2$, $g(x) = 0$, the Eq. (1) becomes classical Lane–Emden type equation. It is also called standard Lane–Emden equation. For special forms of $f(u)$, this equation was used to model numerous phenomena in mathematical physics, thermodynamics, fluid mechanics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres, and theory of thermionic currents [3–5]. The most popular forms of $f(u) = u^{\lambda}$ is used to describe polytrophic and isothermal spheres, having the initial conditions $u(0) = 1$, $u'(0) = 0$.

It is essentially a Poisson's equation adapted to the situations appears in modeling applications such as radiatively cooling, self-gravitating gas clouds, in the mean-field treatment of a phase transition, in critical absorption and clusters of galaxies [6]. Analytical techniques have been considered by various mathematicians and researchers to solve Eq. (1) with $f(u) = u^{\lambda}$, to handle the singularity having index λ. Scientific literature claims that for $0 \leqslant \lambda \leqslant 5$, the solutions of the

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Lane–Emden equation (also called the polytrophic differential equations) could be given in closed form. Goenner and Havas [7] tried to give the exact solution of generalized Lane–Emden equation and Wazwaz [8,9] applied Adomian decomposition method to approximate the same equation in the form of convergent series.

Recently, the idea of homotopy was blended with perturbation for the solutions of nonlinear problems. The fundamental work was done by Liao [10] by introducing Homotopy Analysis Method (HAM) in his Ph.D. dissertations [10]. He introduced Homotopy Perturbation Method (HPM) in 1998 [11]. Variational approach proposed by He, was applied to approximate Lane–Emden type equation in 2003 [12]. Very recently, Marinca et al. [13] established a new technique known as the Optimal Homotopy Asymptotic Method (OHAM). The advantage of OHAM is: built in convergence criteria similar to HAM but more flexible. In series of papers Marinca et al. [14–16] and Iqbal et al. [17] have proved effectiveness, generalization and reliability of the method, and obtained reliable solutions of currently important applications in science and engineering.

In this paper, we articulate the concept of OHAM tenders a reasonable, reliable solution to linear/nonlinear singular initial value problems. To communicate the reliability of the method we deal with different examples in the subsequent section taking into account the homogenous, non-homogenous, linear/nonlinear Lane–Emden type differential equations. OHAM points up its soundness and potential for the solution of mentioned problems in real life applications.

2. OHAM [13–17] Formulation (An Algorithm)

According to the optimal homotopy asymptotic method [13–17,19]:

(a) Write the governing differential equation as:

$$A(u(x)) + h(x) = 0, \quad x \in \Omega, \tag{3}$$

Ω is domain. Now Eq. (3) is decomposed into $A(u) = L(u) + N(u)$, where L is a linear part and N is a nonlinear part but, it is worth noting that this is not necessary. We have a great freedom to choose the so-called auxiliary part L.

(b) Construct an optimal homotopy in an untraditional way $\phi(x;p) : \Omega \times [0,1] \to \mathrm{R}$ which satisfies:

$$(1-p)\{L(\phi(x;p)) + h(x)\} - H(x,p)\{A(\phi(x;p)) + h(x)\} = 0, \tag{4}$$

where $p \in [0,1]$ is an embedding parameter, $H(x,p) = pK_1(x,C_i) + p^2K_2(x,C_i) + \cdots + p^mK_m(x,C_i)$ where C_i ,$i = 1,2,\ldots$ are auxiliary constants. The functions $K_1, K_2, K_3, \ldots, K_m$ reduce to constants for simple problems and for complicated problems these functions depend on x and C_i. The choices of functions $K_m(x,C_i)$ could be exponential, polynomial and soon. It is very important to choose these functions since the convergence of the solution greatly depends on the functions. The auxiliary function $H(x,p)$ provides us with a simple way to adjust and control the convergence also increases the accuracy of the results and effectiveness of the method. The presence of C_i, $i = 1,2,\ldots$ ensures the fast convergence. Eq. (4) is called optimal homotopy equation.

(c) Expand $\phi(x;p,K_i)$ in Taylor's series about p, to get an approximate solution in the following manner,

$$\phi(x;p,K_i) = u_0(x) + \sum_{k=1}^{\infty} u_k(x;K_i)p^k, \quad i = 1,2,\ldots \tag{5}$$

It has been observed that the convergence of the series Eq. (5) depends upon the auxiliary functions or constants K_i. If it is convergent at $p = 1$, one has:

$$\underset{\sim}{u}(x;K_i) = u_0(x) + \sum_{k \geqslant 1} u_k(x;K_i). \tag{6}$$

(d) Substitute Eq. (6) into Eq. (3) results the following expression for residual:

$$R(x;K_i) = L(\underset{\sim}{v}(x;K_i)) + h(x) + N(\underset{\sim}{v}(x;K_i)). \tag{7}$$

If $R(x;K_i) = 0$, then $\underset{\sim}{u}(x;K_i)$ will be the exact solution. Generally it doesn't happen, especially in nonlinear problems. For the determinations of auxiliary constants or functions, K_i, $i = 1,2,\ldots,m$, one can see the [13–17,19].

(e) Put auxiliary constants or functions in Eq. (6), one can get the approximate solution.

Example 1. Non-homogeneous linear Emden–Fowler equation with tropic index $\lambda = 1$.

First we consider the following non homogeneous linear Emden–Fowler equation having tropic index $\lambda = 1$ and $g(x) = x^5 + 30x^3$.

$$\frac{d^2u(x)}{dx^2} = -\frac{2}{x}\left(\frac{du(x)}{dx}\right) - u + g(x), \quad 0 < x \leqslant 1, \tag{8}$$