Applied Mathematics and Computation 217 (2011) 7405–7411

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An approximate analytical solution of time-fractional telegraph equation

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ARTICLE INFO

Keywords:
Telegraph equation
Fractional time derivative
Fractional Brownian motion
Homotopy analysis method

ABSTRACT

In this article, the powerful, easy-to-use and effective approximate analytical mathematical tool like homotopy analysis method (HAM) is used to solve the telegraph equation with fractional time derivative α $(1 < \alpha \leqslant 2)$. By using initial values, the explicit solutions of telegraph equation for different particular cases have been derived. The numerical solutions show that only a few iterations are needed to obtain accurate approximate solutions. The method performs extremely well in terms of efficiency and simplicity to solve this historical model.

1. Introduction

Many phenomena in engineering and applied sciences can be described successfully by developing models using fractional calculus, i.e. the theory of derivatives and integrals of fractional non-integer order [1–4]. Fractional differential equations have gained much attention recently since fractional order system response ultimately converges to the integer order system response. No analytical method was available before 1992 for such equations even for linear fractional differential equations. The homotopy analysis method (HAM) was proposed by Liao [5] to solve fractional differential equations with great success. Recently, Hashim et al. [6] applied the homotopy analysis method (HAM) to fractional initial value problems and showed that HAM is the easy-to-use analytical method and it gives close to the exact solution for both the linear and nonlinear partial differential equations.

The general equation of the second order linear hyperbolic telegraph equation in one-dimension is represented by

$$\frac{\partial^2 u}{\partial t^2} + \alpha \frac{\partial u}{\partial t} + \beta u(x,t) = \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad x \in [a,b], \quad t \in [0,T], \tag{1}$$

where α and β are known constants and $u(x,t)$ is the unknown function. For $\alpha > \beta > 0$, it represents telegraph equation. The equation is commonly used in the study of wave propagation of electric signals in a cable transmission line and also in wave phenomena. The equation is used in modeling reaction–diffusion in various branches of engineering sciences and biological sciences by many researchers like Mohebbi and Dehaghan [7], El-Azab and El-Glamel [8], Yousefi [9], Gao and Chi [10], Dehghan and Ghesmati [11] etc. Eq. (1) represents a damped wave motion for $\alpha > 0$, $\beta = 0$. Recently, Das and Gupta [12] have solved the fractional hyperbolic PDE by using HAM.

In 2007, Atanackovic et al. [13] have analyzed diffusion wave equation with two fractional derivatives of different order on bounded and unbounded spatial domains. In this article both the signaling and Cauchy problems are deduced for different particular cases. Orsingher and Beghin [14] have solved the fundamental solution to time fractional telegraph equations of different kind by using Fourier transform. Huang [15] applied Fourier–Laplace transforms during derivation of the fundamental solution for the Cauchy problem in a whole space domain and signaling problem in a half space domain.

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doi:10.1016/j.amc.2011.02.030

Momani [16] successfully applied ADM to find out the solution of space-time-fractional telegraph equation by considering two different kinds of examples. In 2008, Chen et al. [17] have derived the analytical solution of time fractional telegraph equation for Dirichlet, Neumann and Robin boundary conditions using method of separation of variables. But to the best of the authors' knowledge the general telegraph equation with fractional time derivatives has not yet been solved by using the approximate analytical method HAM.

In 1992, Liao [5] proposed a mathematical tool based on homotopy, a fundamental concept in topology and differential geometry known as Homotopy Analysis Method. It is an analytical approach to get the series solution of linear and nonlinear partial differential equations (PDEs). The difference with the other perturbation methods is that this method is independent of small/large physical parameters. It also provides a simple way to ensure the convergence of series solution. Moreover the method provides great freedom to choose base function to approximate the linear and nonlinear problems [18,19]. Another advantage of the method is that one can construct a continuous mapping of an initial guess approximation to the exact solution of the given problem through an auxiliary linear operator and to ensure the convergence of the series solution an auxiliary parameter is used. Liao and Tan [20] have shown that with the help of there, even complicated nonlinear problems are reduced to the simple linear problems. Recently Liao [21] has claimed that the difference from the other analytical methods is that one can ensure the convergence of series solution by means of choosing a proper value of convergence-control parameter.

In this paper, the homotopy analysis method is used to obtain the approximate analytical solution of telegraph equation with the fractional time derivative for both fractional Brownian motion and standard motion and the results are presented graphically for different particular cases.

2. Solution of the problem

The factional time derivative telegraph equation of order α ($1 < \alpha \leqslant 2$) is given as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial^{\alpha-1}u(x,t)}{\partial t^{\alpha-1}}+u(x,t)=\frac{\partial^2 u(x,t)}{\partial x^2}+f(x,t) \tag{2}$$

with the initial conditions

$$u(x,t)|_{t=0}=0 \quad \text{and} \quad \left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0}=0. \tag{3}$$

To solve Eq. (2) by HAM, we choose the initial approximation

$$u_0(x,t)=0 \tag{4}$$

and the linear operator

$$L[\phi(x,t;q)]=\frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha}, \quad 1<\alpha\leqslant 2 \tag{5}$$

with the property

$$L[C_1+C_2 t]=0, \tag{6}$$

where C_1 and C_2 are integral constants. Furthermore, in view of Eq. (2) we define a differential equation

$$N[\phi(x,t;q)]=\frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha}+\frac{\partial^{\alpha-1}\phi(x,t;q)}{\partial t^{\alpha-1}}+\phi(x,t;q)-\frac{\partial^2 \phi(x,t;q)}{\partial x^2}-f(x,t), \tag{7}$$

where N[•] is the non-linear operator.

Now, we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q)-u_0(x,t)]=q\hbar N[\phi(x,t;q)]. \tag{8}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0)=u_0(x,t) \quad \text{and} \quad \phi(x,t;1)=u(x,t). \tag{9}$$

Therefore, as the embedding parameter q increases from zero to unity, $\phi(x,t;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t;q)$ in Taylor series with respect to q one has

$$\phi(x,t;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)q^m, \tag{10}$$

where

$$u_m(x,t)=\frac{1}{m!}\left.\frac{\partial^m \phi(x,t;q)}{\partial q^m}\right|_{q=0}.$$

If the auxiliary linear operator, the initial guesses and the auxiliary parameter $\hbar$ are properly chosen, the above series is convergent at $q = 1$, and then one has

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t), \tag{11}$$