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New exact explicit peakon and smooth periodic wave solutions of the $K(3,2)$ equation

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ABSTRACT

Based on an independent variable transformation, the $K(3,2)$ equation is investigated by the bifurcation method of planar systems and qualitative theory of polynomial differential system. In different regions of the parametric space, some new exact explicit peakon and smooth periodic wave solutions are obtained. The results presented in this paper improve the previous results.

1. Introduction

To study the role of nonlinear dispersion in the formation of patterns in the liquid drop, Rosenau and Hyman [1] showed that in a particular generalization of the KdV equation:

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m > 0, \quad 1 < n \leqslant 3, \tag{1}$$

which is called $K(m,n)$ equation. They obtained solitary wave solutions with compact support in it, which they called compactons. For the case $m = n$ (m is an integer), these compactons had the property that the width was independent of the amplitude. In Ref. [1], Rosenau and Hyman studied $K(2,2)$ and $K(3,3)$ equations further and they stated that $K(3,2)$ equation had an elliptic function solution. In a later work, Rosenau [2] obtained elliptic function compactons for the cases of $K(4,2)$ and $K(5,3)$. Phase compactons have also been investigated [3]. Rosenau [4] also studied the $K(m,n)$ equation:

$$u_t + a(u^m)_x + (u^n)_{xxx} = 0, \tag{2}$$

where a is a constant. He investigated nonlinear dispersion and compact structures [5], nonanalytic solitary waves [6], and a class of nonlinear dispersive–dissipative interactions [7]. Lately, Ismail and Taha [8] implemented a finite difference method and a finite element method to study the two types of equations $K(2,2)$ and $K(3,3)$. A single compacton as well as the interaction of compactons has been numerically studied. Then, Ismail [9] made an extension to the work in [8] and applied a finite difference method on $K(2,3)$ equation and obtained numerical solutions of $K(2,3)$ equation [10]. Frutos and Lopez-Marcos [11] presented a finite difference method for the numerical integration of $K(2,2)$ equation. Zhou and Tian [12] studied soliton solution of $K(2,2)$ equation. Xu and Tian [13] investigated the peaked wave solutions of $K(2,2)$ equation. Zhou et al. [14] obtained kink-like wave solutions and antikink-like wave solutions of $K(2,2)$ equation. In [15,16], general solutions to the $K(n,n)$ equation were studied. In [17], the nonlinear equation $K(m,n)$ was studied for all possible values of m and n. In [18], Tian and Yin investigated shock-peakon and shock-compacton solutions for $K(m,n)$

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equation by variational iteration method. In 2008, Biswas [19] considered the following $K(m,n)$ equation with generalized evolution term:

$$(u^l)_t + au^m u_x + b(u^n)_{xxx} = 0, \tag{3}$$

he presented a solitary wave ansatz and obtained a 1-soliton solution.

In this paper, we will introduce an independent variable transformation to study the $K(3,2)$ equation:

$$u_t + (u^3)_x + (u^2)_{xxx} = 0, \tag{4}$$

and obtained its some new exact explicit peakon and smooth periodic wave solutions using the bifurcation method of planar systems and qualitative theory of polynomial differential system [20,21].

Using the following independent variable transformation:

$$u(x,t) = \mu + \psi(\xi) = \mu + \psi(x - ct), \tag{5}$$

where c $(c \neq 0)$ is the wave speed, μ is a constant, and substituting (5) into (4), we obtain

$$-c\psi' + ((\mu+\psi)^3)' + ((\mu+\psi)^2)''' = 0, \tag{6}$$

where "$'$" is the derivative with respect to ξ.

Integrating (6) once with respect to ξ and setting the constant of integration to $-\mu^3$, we have the following travelling wave equation of (6):

$$-2(\psi+\mu)\psi'' = \psi^3 + 3\mu\psi^2 + (3\mu^2 - c)\psi + 2(\psi')^2. \tag{7}$$

Letting $y = \frac{d\psi}{d\xi}$, we get the following planar system

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\frac{\psi^3 + 3\mu\psi^2 + (3\mu^2 - c)\psi + 2y^2}{2(\psi+\mu)}. \tag{8}$$

System (8) is a two-parameter planar dynamical system depending on the parameter set (μ, c). Since the phase orbits defined by the vector field of system (8) determine all travelling wave solutions of Eq. (4), we should investigate the bifurcations of phase portraits of system (8) in (ψ, y)-phase plane as the parameters μ, c are changed.

Clearly, on such straight line $\psi = -\mu$ in the phase plane (ψ, y), system (8) is discontinuous. Such system is called a singular travelling wave system by one of authors [22].

The rest of this paper is organized as follows: In Section 2, we discuss the bifurcations of phase portraits of system (8), where explicit parametric conditions will be derived. In Section 3, we give some exact parametric representations of peakon and smooth periodic wave solutions of Eq. (4) in explicit form. A short conclusion will be given in Section 4.

2. Bifurcation sets and phase portraits of system (8)

Using the transformation $d\xi = -2(\psi+\mu)d\tau$, it carries (8) into the Hamiltonian system

$$\frac{d\psi}{d\tau} = -2(\psi+\mu)y, \quad \frac{dy}{d\tau} = \psi^3 + 3\mu\psi^2 + (3\mu^2 - c)\psi + 2y^2. \tag{9}$$

Since both system (8) and (9) have the same first integral

$$(\psi+\mu)^2 y^2 + \left(\frac{1}{5}\psi^3 + \mu\psi^2 + \frac{1}{3}(6\mu^2 - c)\psi + \frac{1}{2}\mu(3\mu^2 - c)\right)\psi^2 = h, \tag{10}$$

then the two systems above have the same topological phase portraits except the line $\psi = -\mu$. Obviously, $\psi = -\mu$ is an invariant straight-line solution of system (9).

Write $\Delta_1 = 4c - 3\mu^2$, $\Delta_2 = \frac{1}{2}\mu(\mu^2 - c)$, $\psi_s = -\mu$. Clearly, when $\Delta_1 > 0$, system (9) has three equilibrium point at $O(0,0), A_{1,2}(\psi_{1,2}, 0)$ in ψ-axis, where $\psi_{1,2} = \frac{-3\mu \pm \sqrt{\Delta_1}}{2}$. When $\Delta_1 = 0$, system (9) has two equilibrium points at $O(0,0)$, $A_{12}(\psi_{12}, 0)$ in ψ-axis, where $\psi_{12} = -\frac{3\mu}{2}$. When $\Delta_1 < 0$, system (9) has only one equilibrium point at $O(0,0)$ in ψ-axis. When $\Delta_2 > 0$, there exist two equilibrium points of system (9) in line $\psi = \psi_s$ at $S_\pm(\psi_s, Y_\pm)$, $Y_\pm = \pm\sqrt{\Delta_2}$.

Let $M(\psi_e, y_e)$ be the coefficient matrix of the linearized system of the system (9) at an equilibrium point (ψ_e, y_e). Then we have

$$M(\psi_e, y_e) = \begin{pmatrix} -2y_e & -2(\psi_e+\mu) \\ 3\psi_e^2 + 6\mu\psi_e + (3\mu^2 - c) & 4y_e \end{pmatrix},$$

and at this equilibrium point, we have

$$Trace(M(\phi_i, y_e)) = 2y_e,$$
$$J(\psi_e, y_e) = detM(\psi_e, y_e) = -8y_e^2 + 6\psi_e^3 + 18\mu\psi_e^2 + 2(9\mu^2 - c)\psi_e + 2\mu(3\mu^2 - c).$$