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Bifurcations of traveling wave solutions for the magma equation

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ABSTRACT

The traveling wave solutions of the magma equation are studied by using the approach of dynamical systems and the theory of bifurcations. With the aid of Maple, all bifurcations and phase portraits in the parametric space are obtained. Under different regions of parametric space, various sufficient conditions to guarantee the existence of solitary wave, periodic wave and breaking wave solutions are given. Moreover, the reason for appearance of breaking waves is explained.

1. Introduction

Many nonlinear partial differential equations(NPDEs) admit traveling wave solutions that move at a constant speed without changing their shapes. Such traveling waves occur in many fields, including biology, chemistry, fluid dynamics and optics. The investigation of exact traveling wave solutions to NPDEs plays an important role in the study of nonlinear physical phenomena. In the past several decades, great progress have been made on the construction of exact solutions of NPDEs and many significant methods have been established such as inverse scattering transformation (IST), bilinear method, symmetry reduction, Bäcklund and Darboux transformations and so on [1–5]. One of powerful methods is the bifurcation theory method of dynamical systems which has been developed to the study of traveling wave solutions of NPDEs [6–11].

In this paper, we consider the magma equation which describes the motion of melt in the Earth. The buoyancy force owing to the density difference of the liquid phase of melt and the solid phase of matrix causes the melt in the earth's mantle to propagate through the partially molten rock. This flow of melt is like a porous flow. Assuming that the liquid phase of melt and the solid phase of matrix are fully connected and incompressible, neglecting the phase transition and allowing only vertical motions, Scott and Stevenson [12] proposed an equation

$$u_t = [u^n(u^{-m}u_t)_x - u^n]_x, \tag{1.1}$$

where x is the vertical space coordinate and t is the time and $u(x,t)$ is the mean volume fraction of the liquid phase which should be nonnegative for any x and t. The exponents n and m denote the dependence of permeability and effective viscosity. It is suggested that the reasonable values of n and m are $0 \leqslant m \leqslant 1$ and $2 \leqslant n \leqslant 5$, respectively. Eq. (1.1) is well-known to have solitary wave solutions and has been examined by various authors [13–21]. Barcilon and Richter [13] solved Eq. (1.1) numerically to obtain solitary wave profiles and considered solitary wave interaction. Takahashi and Statsuma [14] obtained some explicit solutions of Eq. (1.1). Banerjee and Chatterjee [15] investigated the implicit solitary wave solutions of Eq. (1.1). Harris and Clarkson [16] investigated the painlevé property and obtained some solitary wave solutions of Eq. (1.1). Marchant [17] studied the approximate solutions of Eq. (1.1). Krishnan and Yan [18] obtained some periodic solutions of Eq.

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(1.1) by using sinh-Gordon expansion method. Simpson et al. [19] indicated that Eq. (1.1) behaves analogously to the KdV equation in the emerging of solitary waves and solitons. Khamitova [20] investigated the nonlocal conservation laws of Eq. (1.1). Abourabia et al. [21] obtained some analytical solutions of Eq. (1.1). We shall investigate the bifurcations of traveling wave solutions of Eq. (1.1).

Let $u(x,t) = u(x - ct) = u(\xi)$, where c is the propagating velocity. Substituting the traveling wave variable into (1.1) and integrating once, yields

$$cu + cmu^{n-m-1}(u')^2 - cu^{n-m}u'' - u^n + g = 0, \tag{1.2}$$

where "′" is the derivative with respect to ξ and g is integral constant. Eq. (1.2) is equivalent to the planar system

$$\frac{du}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{1}{cu^{n-m}}(cmu^{n-m-1}y^2 - u^n + cu + g). \tag{1.3}$$

System (1.3) is a four-parameter planar dynamical system depending on the parameter set (m, n, c, g). Since the phase orbits defined by the vector fields of (1.3) determine all the traveling wave solutions of (1.1), we should investigate the bifurcations of phase portraits of (1.3) in the (u, y) phase plane as the parameters are changed. Here we consider a physical model where only bounded traveling waves are meaningful. So we only pay attention to the bounded solutions of (1.3).

Suppose that $u(\xi)$ is a continuous solution of (1.3) for $\xi \in \mathbf{R}$ and $\lim_{\xi\to\pm\infty} u(\xi) = \phi_\pm$. It is well-known that (i) $u(\xi)$ is called a solitary wave solution if $\phi_+ = \phi_-$; (ii) $u(\xi)$ is called a kink (or anti-kink) wave solution if $\phi_+ \neq \phi_-$. Usually, a homoclinic, heteroclinic and periodic orbit of (1.3), respectively, corresponds to a solitary, kink and periodic wave of (1.1). Thus, to investigate all possible bifurcations of solitary, kink (or anti-kink) and periodic waves of (1.1), we need to find all periodic annuli, homoclinic orbits and heteroclinic orbits of (1.3), which depend on the system parameters. The bifurcation theory of dynamical systems plays an important role in our study.

We notice that the right hand of the second equation in (1.3) is discontinuous when $u = 0$. In other words, on the above straight line of the phase (u, y), u'' has no definition. It implies that (1.1) has non-smooth traveling wave solutions. We claim that the existence of a singular line for a traveling wave equation is the original reason why traveling waves lose their smoothness (i.e. analytic behavior).

The rest of this paper is organized as follows. In Section 2, we discuss the bifurcations of phase portraits of (2.1). In Section 3, we consider the existence of smooth solitary, kink periodic eaves and non-smooth waves (breaking waves) of Eq. (1.1) and obtain the sufficient conditions to guarantee the existence of the above solutions. And some parametric representations of smooth and non-smooth traveling wave solutions of (1.1) in the different parameter regions are given by using the elliptic functions and hyperbolic functions [22]. In Section 4, we give the summary.

2. Phase portraits and bifurcation sets of Eq. (2.1)

In this section, we shall study all phase portraits and bifurcations sets of (1.3) in the parameter space. Making the "time scale" transformation $d\xi = u^{n-m}d\zeta$, singular system (1.3) becomes the regular system

$$\frac{du}{d\zeta} = u^{n-m}y, \quad \frac{dy}{d\zeta} = mu^{n-m-1}y^2 - \frac{u^n}{c} + u + \frac{g}{c}, \tag{2.1}$$

which has the same topological phase portraits as (1.3) except for the straight line $u = 0$.

For the distribution of equilibrium points on the u-axis, assume that $(u^*, 0)$ be the equilibrium point of Eq. (2.1), then $f(u^*) = 0$ where $f(u) = u^n - cu - g$. Since $f(u)$ has at most three different real roots, Eq. (2.1) also has at most three equilibrium points. Hence if Eq. (2.1) has one or three equilibrium points on the u-axis we denote them, respectively, by E_1 or $E_i(u_i, 0)$ $(i = 1, 2, 3)$ where u_i are real roots of $f(u)$. In addition, when $m = 1,\ n = 2$ and $cg < 0$, system (2.1) has two equilibrium points on the y-axis: $Y_\pm(0, \pm\sqrt{-g/c})$.

Let $M(u_e, y_e)$ be the coefficient matrix of the linearized system of (2.1) at an equilibrium point (u_e, y_e) and $J(u_e, y_e)$ be its Jacobian determinant. By the theory of planar dynamical systems, we know that for an equilibrium point of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and Trace $(M(u_e, y_e)) = 0$ then it is a center point; if $J > 0$ and Trace $(M(u_e, y_e))^2 - 4J(u_e, y_e) > 0$ then it is a node; if $J = 0$ and the Poincare index of the equilibrium point is 0 then it is a cusp. We have

$$J(u_i, 0) = u_i^{n-m}\left(\frac{nu_i^{n-1}}{c} - 1\right), \quad \text{Trace}(M(u_i, 0)) = 0,$$

$$J\left(0, \pm\sqrt{\frac{-g}{c}}\right) = \frac{-2g}{c}, \quad \text{Trace}\left(M\left(0, \pm\sqrt{\frac{-g}{c}}\right)\right) = \pm 3\sqrt{\frac{-g}{c}}.$$

From above facts to do qualitative analysis, we now consider the bifurcations of phase portraits of (2.1) in the case of $m = 0,\ n \geqslant 2\ (n \in Z^+)$ and $m = 1,\ n \geqslant 2$, respectively.