Applied Mathematics and Computation 215 (2010) 4216–4220

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Eigenparameter dependent discrete Dirac equations with spectral singularities

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ARTICLE INFO

Keywords:
Discrete Dirac equations
Spectral analysis
Discrete spectrum
Spectral singularities

ABSTRACT

Let us consider the Boundary Value Problem (BVP) for the discrete Dirac Equations

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)} \end{cases}, n \in \mathbb{N}, \tag{0.1}$$

$$(\gamma_0 + \gamma_1\lambda)y_1^{(2)} + (\beta_0 + \beta_1\lambda)y_0^{(1)} = 0, \tag{0.2}$$

where $(a_n), (b_n), (p_n)$ and $(q_n), n \in \mathbb{N}$ are complex sequences, $\gamma_i, \beta_i \in \mathbb{C}, i = 0, 1$ and λ is a eigenparameter. Discussing the eigenvalues and the spectral singularities, we prove that the BVP (0.1), (0.2) has a finite number of eigenvalues and spectral singularities with a finite multiplicities, if

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta})(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty,$$

holds, for some $\varepsilon > 0$ and $\frac{1}{2} \leqslant \delta \leqslant 1$.

1. Introduction

Discrete boundary value problems have been intensively studied in the last decade. The modeling of certain linear and nonlinear problems from economies, optimal control theory and other areas of study have led to the rapid development of the theory of difference equations. Also the spectral analysis of the difference equations have been treated by various authors in connection with the classical moment problem (see the monographs of Agarwal [5], Agarwal-Wong [8] and Kelley-Peterson [14] and the papers of Agarwal-Perera-O'Regan [6,7] and the references there in). The spectral theory of the difference equations have also been applied to the solution of classes of nonlinear discrete Korteveg–de Vriez equations and Toda lattices ([12,17]).

Let us consider the discrete boundary value problem (BVP):

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{N} = \{1, 2, \ldots\} \tag{1.1}$$

$$y_0 = 0, \tag{1.2}$$

where (a_n) and (b_n) are complex sequences, $a_0 \neq 0$ and λ is a spectral parameter. The spectral analysis of the BVP (1.1), (1.2) with continuous and point spectrum has been studied in [9]. In this article, the authors proved that the spectrum of the BVP (1.1), (1.2) consists of the continuous spectrum, the eigenvalues and the spectral singularities. The spectral singularities are

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doi:10.1016/j.amc.2009.12.043

poles of the resolvent and are also imbedded in the continuous spectrum, but they are not eigenvalues. The effect of the spectral singularities in the spectral expansion of the BVP (1.1), (1.2) in terms of the principal vectors has been investigated in [15]. In [1,2], the dependence of the structure of the spectral singularities of the BVP (1.1), (1.2) on the behaviour of the sequences (a_n) and (b_n) was considered. Some problems related to the spectral analysis of difference equations with spectral singularities have been discussed in [3,4,10,11].

Let us consider the non-selfadjoint BVP for the system of difference equations of first order

$$\begin{cases} a_{n+1}y_{n+1}^{(2)} + b_n y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ a_{n-1}y_{n-1}^{(1)} + b_n y_n^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \quad n \in \mathbb{N}, \end{cases} \tag{1.3}$$

$$(\gamma_0 + \gamma_1\lambda)y_1^{(2)} + (\beta_0 + \beta_1\lambda)y_0^{(1)} = 0, \gamma_0\beta_1 - \gamma_1\beta_0 \neq 0, \tag{1.4}$$

where $\begin{pmatrix} y_n^{(1)} \\ y_n^{(2)} \end{pmatrix}, n \in \mathbb{N}$ are vector sequences, $a_n \neq 0, b_n \neq 0$ for all n, $\gamma_i, \beta_i \in \mathbb{C}, i = 0,1$ and λ is a spectral parameter. If for all $n \in \mathbb{N}, a_n \equiv 1$ and $b_n \equiv -1$ then the system (1.3) reduces to

$$\begin{cases} \Delta y_n^{(2)} + p_n y_n^{(1)} = \lambda y_n^{(1)} \\ -\Delta y_{n-1}^{(1)} + q_n y_n^{(2)} = \lambda y_n^{(2)}, \quad n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where Δ is a forward difference operator, i.e., $\Delta u_n = u_{n+1} - u_n$. The system (1.5) is the discrete analogue of the well-known Dirac system

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & 0 \\ 0 & q(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

([16], Chap. 2). Therefore the system (1.5) (also (1.3)) is called the discrete Dirac system. The specific features of this paper are the presence of the spectral parameter not only in the difference equation and also in the boundary condition.

In this paper, we aim to investigate of eigenvalues and spectral singularities of the BVP (1.3), (1.4) has a finite number of eigenvalues and spectral singularities with a finite multiplicities, if the condition

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta})(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty,$$

holds, for some $\varepsilon > 0$ and $\frac{1}{2} \leqslant \delta \leqslant 1$.

2. Jost solution of (1.3)

Let for some $\varepsilon > 0$ and $\frac{1}{2} \leqslant \delta \leqslant 1$

$$\sum_{n=1}^{\infty} \exp(\varepsilon n^{\delta})(|1 - a_n| + |1 + b_n| + |p_n| + |q_n|) < \infty \tag{2.1}$$

satisfied. It is well-known that [11], under the condition (2.1). Eq. (1.3) has the bounded solution

$$f_n(z) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix} = \alpha_n\left(I_2 + \sum_{m=1}^{\infty} A_{nm}e^{imz}\right)\begin{pmatrix} e^{i\frac{z}{2}} \\ -i \end{pmatrix}e^{inz}, \quad n \in \mathbb{N}, \tag{2.2}$$

$$f_0^{(1)}(z) = \alpha_0^{11}\left\{e^{i\frac{z}{2}}\left[1 + \sum_{m=1}^{\infty} A_{0m}^{11}e^{imz}\right] - i\sum_{m=1}^{\infty} A_{0m}^{12}e^{imz}\right\}, \tag{2.3}$$

for $\lambda = 2\sin\frac{z}{2}$ and $z \in \overline{\mathbb{C}}_+ := \{z : z \in \mathbb{C}, \mathrm{Im} z \geqslant 0\}$, where

$$\alpha_n = \begin{pmatrix} \alpha_n^{11} & \alpha_n^{12} \\ \alpha_n^{21} & \alpha_n^{22} \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_{nm} = \begin{pmatrix} A_{nm}^{11} & A_{nm}^{12} \\ A_{nm}^{21} & A_{nm}^{22} \end{pmatrix}.$$

Note that α_n^{ij} and $A_{nm}^{ij} (i,j = 1,2)$ are expressed in terms of $(a_n), (b_n), (p_n)$ and $(q_n), n \in \mathbb{N}$. Moreover

$$|A_{nm}^{ij}| \leqslant C \sum_{k=n+\left[\left|\frac{m}{2}\right|\right]}^{\infty} (|1 - a_k| + |1 + b_k| + |p_k| + |q_k|), \tag{2.4}$$

holds, where $\left[\left|\frac{m}{2}\right|\right]$ is the integer part of $\frac{m}{2}$ and $C > 0$ is a constant. Therefore f_n is vector-valued analytic function with respect to z in $\mathbb{C}_+ := \{z : z \in \mathbb{C}, \mathrm{Im} z > 0\}$ and continuous in $\overline{\mathbb{C}}_+$ ([11]). The solution $f(z) = (f_n(z)) = \begin{pmatrix} f_n^{(1)}(z) \\ f_n^{(2)}(z) \end{pmatrix}$ is called Jost solution of (1.3).