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A new family of k-Fibonacci numbers

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ABSTRACT

In the present paper, we give a new family of k-Fibonacci numbers and establish some properties of the relation to the ordinary Fibonacci numbers. Furthermore, we describe the recurrence relations and the generating functions of the new family for $k = 2$ and $k = 3$, and presents a few identity formulas for the family and the ordinary Fibonacci numbers.

1. Introduction and objectives

Fibonacci numbers are one of the most well-known numbers, and it has many important applications to diverse fields such as mathematics, computer science, physics, biology, and statistics. For the history of Fibonacci numbers, see a very recent book [15]. For the applications, see, e.g. [2,4]. For the theory, see, e.g. [9,13,23].

There are many important generalizations of Fibonacci numbers, e.g. [13, Chap. 7,1,3,8,10–12,14,16–19,21,22,24]. Among the generalizations, the k-Fibonacci numbers have been recently considered by many authors, see, e.g. [5–7,19,24]. In the present paper, we shall focus on a new family of these numbers.

The rest of this section gives a brief review on the Fibonacci numbers and defines the new family of the k-Fibonacci numbers. In Section 2, we establish some properties of the family. In Section 3, we make some concluding remarks.

It is well known that the Fibonacci numbers: F_n for $n = 0, 1, \ldots$ are defined by the Binet's formula as follows:

$$F_n := \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}), \quad n = 0, 1, \ldots, \tag{1}$$

where $\alpha := \left(1 + \sqrt{5}\right)/2$ and $\beta := \left(1 - \sqrt{5}\right)/2$. The first few Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$ For more detail, one may use the well-known on-line encyclopedia of integer sequences [20]. The numbers F_n satisfy the second order linear recurrence relation:

$$F_n = F_{n-1} + F_{n-2}, \quad n = 1, 2, \ldots \tag{2}$$

with the initial conditions $F_{-1} = 0,\ F_0 = 1$. It is also widely known that the F_n is related by the determinant of the special tridiagonal matrix of the form

$$T_n = \begin{pmatrix} 1 & 1 & & & \\ -1 & 1 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 1 \\ & & & -1 & 1 \end{pmatrix} \in R^{n\times n}. \tag{3}$$

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The Lucas numbers L_n are closely related to the Fibonacci numbers F_n. The Lucas numbers are defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n = 2, 3, \ldots$$

with initial conditions $L_0 = 2$ and $L_1 = 1$. The first few Lucas numbers are $2, 1, 3, 4, 7, 11, 18, 29, 47, 76, \ldots$ The Binet's formula for the Lucas numbers L_n is $L_n = \alpha^n + \beta^n,\ n = 0, 1, \ldots$ Then, we see that the Lucas and Fibonacci numbers are related by

$$L_n = F_n + F_{n-2} = \frac{F_{2n-1}}{F_{n-1}}.$$

Now, we define a new generalized k-Fibonacci number as follows.

Definition 1.1 (*k-Fibonacci numbers*). Let n and k ($\neq 0$) be natural numbers, then there exist unique numbers m and r such that $n = mk + r$ ($0 \leqslant r < k$). Using these parameters, we define generalized k-Fibonacci numbers $F_n^{(k)}$ by

$$F_n^{(k)} := \frac{1}{\left(\sqrt{5}\right)^k}(\alpha^{m+2} - \beta^{m+2})^r(\alpha^{m+1} - \beta^{m+1})^{k-r}, \quad n = mk + r. \tag{4}$$

The first few numbers of the new family for $k = 2, 3$ are as follows:

$$\left\{F_n^{(2)}\right\}_{n=0}^{10} = \{1, 1, 1, 2, 4, 6, 9, 15, 25, 40, 64\},$$

$$\left\{F_n^{(3)}\right\}_{n=0}^{10} = \{1, 1, 1, 1, 2, 4, 8, 12, 18, 27, 45\}.$$

For the other numbers, see Table 1 in Appendix A.

From (1) and Definition 1.1, the generalized k-Fibonacci and Fibonacci numbers are related by

$$F_n^{(k)} = (F_m)^{k-r}(F_{m+1})^r, \quad n = mk + r. \tag{5}$$

Considering the case $k = 1$ in (4), we see that $m = n$ and $r = 0$. Therefore, $F_n^{(1)}$ is the ordinary Fibonacci numbers F_n.

In the next section, we give some properties of the generalized k-Fibonacci numbers.

2. Main results

Theorem 2.1. *Let $k,\ m \in \{1, 2, 3, \ldots\}$.*

For fixed numbers $k,\ m$, the generalized k-Fibonacci numbers and the ordinary Fibonacci numbers satisfy

(i) $\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}F_{mk+i}^{(k)} = (-1)^{k-1}F_mF_{(m-1)(k-1)}^{(k-1)}$.

(ii) $\sum_{i=0}^{k-1}\binom{k-1}{i}F_{mk+i}^{(k)} = F_mF_{(m+2)(k-1)}^{(k-1)} = F_m(F_{m+2})^{k-1}$.

(iii) $\sum_{i=0}^{k-1}F_{mk+i}^{(k)} = \frac{F_m}{F_{m-1}}\left[(F_{m+1})^k - (F_m)^k\right] = \frac{F_m}{F_{m-1}}\left[F_{(m+1)k}^{(k)} - F_{mk}^{(k)}\right]$.

Proof.

(i) We have

$$\begin{aligned}
\sum_{i=0}^{k-1}(-1)^i\binom{k-1}{i}F_{mk+i}^{(k)} &= (-1)^{k-1}\sum_{i=0}^{k-1}(-1)^{k-1-i}\binom{k-1}{i}(F_m)^{k-i}(F_{m+1})^i \quad (\text{from } (5)) \\
&= (-1)^{k-1}F_m\sum_{i=0}^{k-1}\binom{k-1}{i}(F_{m+1})^i(-F_m)^{k-1-i} \\
&= (-1)^{k-1}F_m(F_{m+1} - F_m)^{k-1} \quad (\text{by using the binomial theorem}) \\
&= (-1)^{k-1}F_m(F_{m-1})^{k-1} \quad (\text{from } (2)) \\
&= (-1)^{k-1}F_mF_{(m-1)(k-1)}^{(k-1)} \quad (\text{from } (5) \text{ with } r = 0).
\end{aligned}$$