Applied Mathematics and Computation 216 (2010) 1375–1382

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the solution of a class of the reduced wave equation and the Riccati differential equation

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ARTICLE INFO

Keywords:
Differential equation
Riccati differential equation
Acoustic scattering problem
Helmholtz equation

ABSTRACT

In this paper we present an "iteration" technique for a class of differential equation having the form $z'' = \lambda z$, where λ is a function in C^∞. We show that we can construct not only the general solution of the reduced wave equation but also the general solution of the Riccati differential equation by using this iteration technique if the given function λ is satisfies the condition

$$\frac{\lambda_{2n-1}}{\lambda_{2n-2}} = \frac{\lambda_{2n-3}}{\lambda_{2n-4}} := \beta.$$

Then we give a simple application of this technique to inverse conductive scattering problem for an inhomogeneous spherical medium.

1. Introduction

The wave equation is an important partial differential equation that describes a variety of waves, such as sound, light and water waves. It arises in acoustic, electromagnetic and fluid dynamics.

The propagation of waves in a homogeneous, isotropic medium is mathematically described by the wave equation.

$$\Delta V(x,t) - \frac{1}{c^2} V_{tt}(x,t) = 0, \tag{1.1}$$

where Δ is the Laplace operator and c denotes the speed of propagation. If the problems involve the time-harmonic waves, i.e. wave fields of the form

$$V(x,t) = u(x)e^{-iwt}, \tag{1.2}$$

where $i = \sqrt{-1}$ and w denotes the speed of propagation, then the wave equation can be reduced to the homogeneous scalar Helmholtz equation

$$\Delta u(x) + k^2 u(x) = 0, \quad k = \frac{w}{c}. \tag{1.3}$$

Some particular solution of Eq. (1.3) in $\mathbb{R}^n (n = 1, 2, 3)$ have been investigated in [3,17,18].

If we consider an inhomogeneous absorbing medium in $\mathbb{R}^3$ and if we assume that the inhomogeneity is compactly supported, the propagation of time-harmonic acoustic waves in the medium is governed by the equation

$$\Delta u(x) + k^2 n(x)u(x) = 0, \quad x \in \mathbb{R}^3, \tag{1.4}$$

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doi:10.1016/j.amc.2010.03.008

where u describes the pressure field, $k > 0$ is the wave number and $n(x) = n_1(x) + i\frac{n_2(x)}{k}$ is the refractive index of the medium [6,16,19].

Elementary solution of the reduced wave equation for variable index of refraction n is well-known (point source) $u = \frac{\exp(ikR)}{R}$ where $R^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$ and (line source) $u = i\pi H_0^{(1)}(kQ)$, where $Q^2 = (x - x_0)^2 + (y - y_0)^2$ and $H_0^{(1)}$ is the Hankel function of the first kind of order zero. If the medium is homogeneous ($n = 1$) the solutions are known to be very few in number. Additionally, for layered media $[n = n(y)]$ Pekeris' solution [15] for a point source in a medium specified by $n = y^{-1}$ is

$$u = \frac{2(yy_0)^{\frac{1}{2}}}{RR'} \exp\left[2i\left(k^2 - \frac{1}{4}\right)^{\frac{1}{2}} \tanh^{-1}\left(\frac{R}{R'}\right)\right], \tag{1.5}$$

with $R'^2 = (x - x_0)^2 + (y + y_0)^2 + (z - z_0)^2$; and Kormilitsin's solution [12] for a line source extending parallel to the z axis in a medium specified by $n = y^{\frac{1}{2}}$ is

$$u = \int_0^\infty \exp\left[ik\left(\frac{Q^2}{2\zeta} + (y + y_0)\frac{\zeta}{4} - \frac{\zeta^3}{96}\right)\right]\frac{d\zeta}{\zeta}. \tag{1.6}$$

Holford [11] has discussed the elementary solution of reduced wave equation in two dimensions for which the refraction index is the form $n = (A + Bx + +Cy + Dx^2 + Exy + Fy^2)^{\frac{1}{2}}$.

Colton and Wendland [7] have discussed the exterior Neumann problem for the reduced wave equation in a spherical symmetric medium. Li et al. [13] have presented an exact solution of the Helmholtz equation for $n = (1 + Az)^{\frac{1}{2}}$. Hilton [10] has studied the Strum–Liouville solution of the 3D-wave equation when the refractive index depends only on the radial variable $r = |x|$. Chomorro et al. [4] have obtained the exact solution of the nonlinear Helmholtz equation by a lucid generalization of paraxial solution theory.

However, in recent years a lot of useful numerical methods and solutions have been presented for the reduced wave equation with variable coefficient [1,4,8,12,13].

The main purpose of this article is to present a new method for the solution of Eq. (1.4) and the Riccati differential equation.

The rest of this article is organized as follows: in Section 2, we introduce a model solution for Eq. (1.4) and we shall give a new technique, which can be called the asymptotic iteration method [5], to solve a second-order linear differential equations of the form $z'' = \lambda_0(r)z$, where $\lambda_0 \in C^\infty(0, r)$. In Section 3, we give an application for the inverse conductive scattering problem by the solution of Eq. (1.4) with new method. Finally in Section 4 we present some remarks.

2. Analytic solution of the reduced wave equation

Let $R = R(x, y) = |x - y|$ denote the distance between two typical points x and y in $\mathbb{R}^3$. A fundamental solution of the scalar Helmholtz equation $\Delta u + k^2 u = 0, k > 0$, is a two point function of position $\Phi(x, y)$ which for convenience can be written in the form

$$\Phi(x, y) = -\frac{1}{4\pi}\frac{e^{ikR}}{R}, \quad R \neq 0. \tag{2.1}$$

If we choose that one of these two points fixed then the fundamental solution $\Phi(x, y)$ becomes a one variable function. Let y be fixed point, then $\Phi(x, y)$ is a solution of the scalar Helmholtz equation in $\mathbb{R}^3 \setminus \{y\}$. If we choose the point y as the origin then $R = |x|$ and we denote as usual $r = |x|$. Then the fundamental solution of the scalar Helmholtz equation can be considered as a radial function. In addition, we know that the real and imaginary parts of the fundamental solution are also a solution of the scalar Helmholtz equation.

In the rest of this paper, we assume that $n(x) = n(|x|) = n(r)$ and denote

$$P(r) := k^2 - k^2 n(r). \tag{2.2}$$

Lemma 1. *Let $f : \mathbb{R}^3 \to \mathbb{R} \setminus \{0\}$ be a continuous function that has first and second derivatives and satisfies the equation*

$$P(r) = -\frac{f''(r)}{f(r)} + 2\frac{(f'(r))^2}{f^2(r)} - (2k\cot kr)\frac{f'(r)}{f(r)} \tag{2.3}$$

then

$$u(x) = \frac{\sin kr}{rf(r)}, \quad r = |x| \tag{2.4}$$

satisfies Eq. (1.4).

Note. In the rest of this paper, for the sake of simplicity we use only f and P instead of $f(r)$ and $P(r)$.