Applied Mathematics and Computation 216 (2010) 1624–1636

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

An inexact parallel splitting augmented Lagrangian method for large system of linear equations

Zheng Peng [a,b,*], DongHua Wu [b]

[a] Mathematics Department of Nanjing University, Nanjing 210093, PR China
[b] Mathematics Department of Shanghai University, Shanghai 200444, PR China

ARTICLE INFO

Keywords:
System of linear equations
Parallel iteration
Matrix decomposition
Augmented Lagrangian method
Inexact terms

ABSTRACT

Parallel iterative methods are powerful in solving large systems of linear equations (LEs). The existing parallel computing research results focus mainly on sparse systems or others with particular structure. Most are based on parallel implementation of the classical relaxation methods such as Gauss–Seidel, SOR, and AOR methods which can be efficiently carried out on multiprocessor system. In this paper, we propose a novel parallel splitting operator method in which we divide the coefficient matrix into two or three parts. Then we convert the original problem (LEs) into a monotone (linear) variational inequality problem (VI) with separable structure. Finally, an inexact parallel splitting augmented Lagrangian method is proposed to solve the variational inequality problem (VI). To avoid dealing with the matrix inverse operator, we introduce proper inexact terms in subproblems such that the complexity of each iteration of the proposed method is $O(n^2)$. In addition, the proposed method does not require any special structure of system of LEs under consideration. Convergence of the proposed methods in dealing with two and three separable operators respectively, is proved. Numerical computations are provided to show the applicability and robustness of the proposed methods.

1. Introduction

Many problems in science and engineering involve the computation of large systems of linear equations for finding solutions. Some of these systems may have millions of variables. Very often, difficulties arise owing to the tremendous size of such problems. Researchers have to resort to self-absorption in parallel computing, a class of methods involving the use of multiprocessor systems simultaneously for finding solutions of a problem. These methods save time in computing a solution of large scale problems.

In this paper, we consider the following well-determined system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in R^{n\times n},\ x \in R^n,\ b \in R^n$, and A matrix is positive definite. A pair of matrices (M, N) with M nonsingular (and easily invertible in practice) such that

$$A = M - N$$

is called a splitting (or regular decomposition) of A.

* Corresponding author. Address: Mathematics Department of Nanjing University, 210093 Nanjing, PR China.
E-mail addresses: pzheng008@gmail.com, pv068@163.com (Z. Peng).

0096-3003/$ - see front matter
doi:10.1016/j.amc.2010.03.018

Based on the splitting (M, N), a general iterative method for solving Problem (1.1) is defined by

$$\begin{cases} x_0 \quad \text{given in } R^n, \\ Mx_{k+1} = Nx_k + b, \quad \forall k \geqslant 1. \end{cases} \tag{1.2}$$

From (1.2) we get

$$x_{k+1} = M^{-1}Nx_k + M^{-1}b. \tag{1.3}$$

Let $J = M^{-1}N$, the iterative method (1.2) and (1.3) converges if and only if the spectral radius of J satisfies $\rho(J) < 1$.

The existing major iterative methods [1–3] are Jacobi method, Gauss–Seidel method and Successive Over-Relaxation (SOR) method. From the practical point of view, the main advantage of Jacobi and Gauss–Seidel method is the avoidance of using the inverse operator M^{-1} in (1.3) in some sense.

Many research results are based on the above iterative methods. For examples, Bagnara [4] gave a unified proof for the convergence of Jacobi and Gauss–Seidel methods, Salkuyeh [5] generalized the Jacobi and Gauss–Seidel methods.

Bai and his co-workers [6,7] studied necessary and sufficient convergence conditions for splitting iteration methods when employed to solve a non-Hermitian system of linear equations with coefficient matrix either a L-matrix or a H-matrix and positive definite, and the convergence conditions for the additive and the multiplicative splitting iteration methods [8], convergence of parallel nonstationary multisplitting iteration methods [9], etc. Bai, Sun and Wang [10] presented a very good review paper about parallel multisplitting methods. In this work the authors gave an unified framework for the construction of various synchronous and asynchronous parallel matrix multisplitting iterative methods. Bai [11–13] also proposed some parallel splitting methods for large systems with various peculiar structure.

In addition, Wang and Huang [14] investigated the convergence and the monotone convergence theories for the alternating method when the coefficient matrix is a H-matrix or a monotone matrix. Hadjidimos [15] established an accelerated over-relaxation method based upon reasonable decomposition of Problem (1.1). Frommer [16] and Neumann [17] obtained convergence results for relaxed parallel multisplitting methods and parallel multisplitting iterative methods for M-matrices.

However, another novel way to solve Problem (1.1) is to convert it into an optimization problem of the form:

$$x^* = \text{Arg} \min_{x \in R^n} \|Ax - b\|_2^2. \tag{1.4}$$

It is well known that Problem (1.4) is identical to the following linear variational inequality problem:

$$\text{Find } x^* \in R^n, \quad \text{such that } (x - x^*)^T \left(A^T(Ax^* - b)\right) \geqslant 0, \quad \forall x \in R^n. \tag{1.5}$$

Recent research shows that projection–contraction methods are powerful tools for solving Problem (1.5). For example, He [18,19] developed a class of projection–contraction (PC) methods for monotone variational inequalities (MVIs), including linear variational inequalities (LVIs).

Furthermore, as mentioned earlier, when A matrix is large, operator splitting and parallel computing will be necessary. On operator splitting and parallel computing for variational inequalities, Glowinski [20], Fukushima [21] and He [22] have made some important contributions.

In [22], the author proposed a parallel splitting augmented Lagrangian method (abbreviate to PSALM) for solving the following variational inequalities:

$$\text{Find } w = \begin{pmatrix} x \\ y \\ \lambda \end{pmatrix} \in \mathcal{W}, \quad \text{such that} \begin{cases} (x' - x)^T \left[f(x) - A^T\lambda\right] \geqslant 0, \\ (y' - y)^T \left[g(y) - B^T\lambda\right] \geqslant 0, \quad \forall w' \in \mathcal{W}, \\ Ax + By - d = 0, \end{cases} \tag{1.6}$$

where $\mathcal{W} = \mathcal{X} \times \mathcal{Y} \times R^m$, $\mathcal{X} \in R^{n_1}$, $\mathcal{Y} \in R^{n_2}$, $A \in R^{m \times n_1}$ and $B \in R^{m \times n_2}$, and $f(x), g(y)$ are given monotone operators with respect to $\mathcal{X}, \mathcal{Y}$ respectively.

For a given (x^k, y^k, λ^k), the PSALM method finds iteratively $\tilde{x}^k, \tilde{y}^k$ by solving the following variational inequalities in a parallel manner:

$$x \in \mathcal{X}, \quad (x' - x)^T \left\{f(x) - A^T\left[\lambda^k - H(Ax + By^k - d)\right]\right\} \geqslant 0, \quad \forall x' \in \mathcal{X}, \tag{1.7}$$

$$y \in \mathcal{Y}, \quad (y' - y)^T \left\{g(y) - B^T\left[\lambda^k - H(Ax^k + By - d)\right]\right\} \geqslant 0, \quad \forall y' \in \mathcal{Y}. \tag{1.8}$$

Then update multiplier by:

$$\tilde{\lambda}^k = \lambda^k - H(A\tilde{x}^k + B\tilde{y}^k - d), \tag{1.9}$$

where H is a given positive matrix. Finally the PSALM method generates the new iterate $w^{k+1} = (x^{k+1}, \ y^{k+1}, \ \lambda^{k+1})$ by letting

$$w^{k+1} = w^k - \alpha^k G^{-1} M(w^k - \tilde{w}^k), \tag{1.10}$$