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On state-dependent delay partial neutral functional integro–differential equations

José P.C. dos Santos

Departamento de Ciências Exatas, Universidade Federal de Alfenas, 37130-000 Alfenas, MG, Brazil

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ABSTRACT

In this paper the existence of mild solutions for a class of abstract neutral integro–differential equations with state-dependent delay is studied.

1. Introduction

In this paper we study the existence of mild solutions for a class of abstract partial neutral integro–differential equations with state-dependent delay described in the form

$$\begin{cases} \frac{d}{dt}(x(t)+f(t,x_t)) = Ax(t) + \int_0^t B(t-s)x(s)ds + g(t,x_{\rho(t,x_t)}), & t \in I = [0,b], \\ x_0 = \varphi \in \mathscr{B}, \end{cases} \tag{1.1}$$

where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X,\ t \geqslant 0$, are closed linear operators; $(X, \|\cdot\|)$ is a Banach space; the history $x_t : (-\infty, 0] \to X$, defined by $x_t(\theta) := x(t+\theta)$ belongs to an abstract phase space $\mathscr{B}$ defined axiomatically and $f, g : I \times \mathscr{B} \to X$ and $\rho : I \times \mathscr{B} \to (-\infty, b]$ are appropriated functions.

Functional differential equations with state-dependent delay appear frequently in applications as model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with first order functional differential equations for which the state belong to some finite dimensional space, see among another works, [1–3,5,6,11–14,23,25,26]. The problem of the existence of solutions for partial functional differential equations with state-dependent delay has been recently treated in the literature in [19,20,18,21]. To the best of our knowledge, the existence of solutions for abstract partial functional integro–differential equations with state-dependent delay is an untreated topic in the literature and this fact is the main motivation of the present work.

2. Preliminaries

In what follows we recall some definitions, notations and results that we need in the sequel. Throughout this paper, $(X, \|\cdot\|)$ is a Banach space and $A, B(t),\ t \geqslant 0$, are closed linear operators defined on a common domain $\mathscr{D}$ which is dense in X. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. In this paper, the notation $\mathscr{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathscr{L}(Z)$ when $Z = W$. The notation, $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. On the other hand, for a bounded function $\gamma : [0, a] \to Z$ and $t \in [0, a]$, the notation $\|\gamma\|_{Z,t}$ is given by

E-mail address: zepaulo@unifal-mg.edu.br

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$$\|\gamma\|_{Z,t} = \sup\{\|\gamma(s)\|_Z : s \in [0,t]\}, \tag{2.1}$$

and we simplify this notation to $\|\gamma\|_t$ when no confusion about the space Z arises.

To obtain our results, we assume that the integro–differential abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds, & t \geqslant 0, \\ x(0) = u_0 \in X, \end{cases} \tag{2.2}$$

has an associated resolvent operator of bounded linear operators $(R(t))_{t\geqslant 0}$ on X.

Definition 2.1. A one-parameter family $(R(t))_{t\geqslant 0}$ of bounded linear operators is a resolvent operator family for (2.2) if the following conditions are verified

(i) $R(0) = I_d$ and $R(\cdot)x \in C([0,\infty);X)$ for every $x \in X$;
(ii) for $x \in D(A)$, $AR(\cdot)x \in C([0,\infty);X)$ and $R(\cdot)x \in C^1([0,\infty);X)$;
(iii) for every $x \in D(A)$ and $t \geqslant 0$,

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)x\,ds, \tag{2.3}$$

$$R'(t)x = R(t)Ax + \int_0^t R(t-s)B(s)x\,ds. \tag{2.4}$$

Let $h \in C([0,b];X)$ and consider the integro–differential abstract Cauchy problem

$$\begin{cases} x'(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + h(t), & t \in [0,b], \\ x(0) = z \in X. \end{cases} \tag{2.5}$$

Motivated by Grimmer [9], we adopt the following concepts of mild solutions for the non-homogeneous system (2.5).

Definition 2.2. A function $u \in C([0,b];X)$ is called a mild solution of (2.5) on $[0,b]$, if $u(0) = z$ and

$$u(t) = R(t)z + \int_0^t R(t-s)h(s)ds, \quad t \in [0,b]. \tag{2.6}$$

For more information on partial integro–differential equations, resolvent of operators and related issues, we refer the reader to [8,4,9,10].

We will herein define the phase space $\mathscr{B}$ axiomatically, using ideas and notations developed in [22]. More precisely, $\mathscr{B}$ will denote the vector space of functions defined from $(-\infty,0]$ into X endowed with a seminorm denoted $\|\cdot\|_{\mathscr{B}}$ and such that the following axioms hold:

(A) If $x : (-\infty, \sigma+b) \to X$, $b > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma+b)$ and $x_\sigma \in \mathscr{B}$, then for every $t \in [\sigma, \sigma+b)$ the following conditions hold:
  (i) x_t is in $\mathscr{B}$.
  (ii) $\|x(t)\| \leqslant H\|x_t\|_{\mathscr{B}}$.
  (iii) $\|x_t\|_{\mathscr{B}} \leqslant K(t-\sigma)\sup\{\|x(s)\| : \sigma \leqslant s \leqslant t\} + M(t-\sigma)\|x_\sigma\|_{\mathscr{B}}$,

where $H > 0$ is a constant; $K, M : [0,\infty) \to [1,\infty)$, $K(\cdot)$ is continuous, $M(\cdot)$ is locally bounded and H, K, M are independent of $x(\cdot)$.

(A1) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma+b)$ into $\mathscr{B}$.
(B) The space $\mathscr{B}$ is complete.

Example 2.1. The phase space $\mathbf{C_r} \times \mathbf{L^p}(\mathbf{g}, \mathbf{X})$.
Let $r \geqslant 0$, $1 \leqslant p < \infty$ and let $g : (-\infty, -r] \to \mathbb{R}$ be a non-negative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [22]. Briefly, this means that g is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty,0]$ such that $g(\xi+\theta) \leqslant \gamma(\xi)g(\theta)$, for all $\xi \leqslant 0$ and $\theta \in (-\infty,-r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty,-r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(g,X)$ consists of all classes of functions $\varphi : (-\infty,0] \to X$ such that φ is continuous on $[-r,0]$, Lebesgue-measurable, and $g\|\varphi\|^p$ is Lebesgue integrable on $(-\infty,-r)$. The seminorm in $C_r \times L^p(g,X)$ is defined by

$$\|\varphi\|_{\mathscr{B}} := \sup\{\|\varphi(\theta)\| : -r \leqslant \theta \leqslant 0\} + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|^p d\theta\right)^{1/p}.$$

The space $\mathscr{B} = C_r \times L^p(g;X)$ satisfies axioms **(A), (A-1), (B).** Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^0 g(\theta)\,d\theta\right)^{1/2}$, for $t \geqslant 0$. (see [22, Theorem 1.3.8] for details).