Applied Mathematics and Computation 216 (2010) 1645–1651

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Perturbation theorems for estimating magnitudes of roots of polynomials

M. Pakdemirli [a,*], N. Elmas [b]

[a] *Department of Mechanical Engineering, Celal Bayar University 45140 Muradiye, Manisa, Turkey*
[b] *Turgutlu Vocational High School, Celal Bayar University, Turgutlu, Manisa, Turkey*

ARTICLE INFO

Keywords:
Roots of polynomials
Perturbation theory

ABSTRACT

Two additional new theorems are posed and proven to estimate the magnitudes of roots of polynomials. Perturbation theory and the order of magnitude of terms are employed to develop the theorems. The theorems may be useful to estimate the order of magnitudes of the roots of a polynomial *a priori* before solving the equation. The theorems are developed for two special classes of polynomials of arbitrary order with their coefficients satisfying certain conditions. Numerical applications of the theorems are presented as examples. It is shown that the theorems produce good estimates for the magnitudes of roots.

1. Introduction

Finding roots of polynomials is a problem of interest for applied oriented researchers. Polynomial type equations frequently appear in many branches of science such as mechanical vibrations, fluid mechanics, heat transfer, strength of materials, economics etc. to name a few. For cubic and higher order polynomials, numerical techniques are used to find the roots. Many different algorithms (Newton–Raphson, Muller, Secant, Householder's Iteration etc.) were already developed to manage this task which was discussed in detail in numerical analysis books (see [1,2] for example). The algorithms are iterative and the convergence to a root requires a good initial estimate.

Recently, two theorems are presented and proven by Pakdemirli and Yurtsever [3] to guide the researcher for a good initial estimate. The theorems are developed based on the order of magnitude concept of the perturbation theory. For details of perturbation theory and applications, the beginner reader is referred to Nayfeh [4] and Hinch [5]. The link between perturbation theory and root finding algorithms were exploited in a series of recent papers [6–8]. Abbasbandy [9] proposed new root finding algorithms using Adomian decomposition theory.

Here, in this paper the main focus is on the initial estimation of the magnitudes of roots and not finding the roots itself. In addition to the two theorems given and proven in [3], two new theorems are posed and proven to estimate the magnitudes of roots. Several numerical applications of the theorems are presented. A good agreement between the predictions of the theories and numerical analysis results are found.

2. Preliminaries and previous work

In perturbation theory, the magnitude of terms are ordered with respect to a small parameter usually expressed as ε, ε being a much smaller quantity than $1 (\varepsilon \ll 1)$. Therefore a term of order $1/\varepsilon$, denoted by $O(1/\varepsilon)$ is much bigger than 1.

* Corresponding author.
E-mail address: mpak@bayar.edu.tr (M. Pakdemirli).

0096-3003/$ - see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2010.03.020

Depending on the magnitudes of coefficients of polynomials, two classes of polynomial equations were defined and theorems for estimating the magnitudes of roots were given previously.

2.1. Polynomial with all coefficients the same order of magnitude

The below theorem was given in [3].

Theorem 1. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0 = 0, \tag{1}$$

if all coefficients a_i $(i = 0 \ldots n)$ are of the same order of magnitude, then the magnitude of roots are of O(1).

Proof. See [3] for details. □

2.2. Polynomial with one relatively large coefficient

For a polynomial equation in which one coefficient is substantially larger than the others with all the remaining coefficients being of order one, the below theorem was given in [3]:

Theorem 2. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_m x^m + \cdots + a_1 x + a_0 = 0, \tag{2}$$

if $a_m \sim O\left(\frac{1}{\varepsilon^k}\right)$ $(k > 0)$ with all other coefficients being of O(1), then the possible roots may be of either $O(\varepsilon^{k/m})$ $(m \neq 0$ case$)$ or $O\left(\frac{1}{\varepsilon^{k/(n-m)}}\right)$ $(m \neq n$ case$)$.

Proof. See [3] for details. □

3. New theorems

Two additional new theorems are stated and proven in this section. The first theorem is about a polynomial equation with one relatively small coefficient and the second theorem is about a polynomial with two relatively large coefficients.

3.1. Polynomial with one relatively small coefficient

If one of the coefficients of a polynomial equation is much smaller than the others which are of the same order, the following theorem is stated:

Theorem 3. *For the polynomial equation*

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_m x^m + \cdots + a_1 x + a_0 = 0, \tag{3}$$

if all coefficients are O(1) except a_m which is much smaller, i.e. $a_m \sim O(\varepsilon^k), \varepsilon \ll 1, k > 0$ *then the magnitudes of roots are*

(i) $x \sim O(1)$ *for m arbitrary,*
(ii) $x \sim O\left(\frac{1}{\varepsilon^k}\right)$ *if m = n,*
(iii) $x \sim O(\varepsilon^k)$ *if m = 0.*

Proof. Three cases for magnitudes of roots are possible: roots may be $O(1)$, small or large.

(i) $x \sim O(1)$:
If x is of order one, then each term in Eq. (3) can be written in their orders of magnitudes

$$O(1) + O(1) + \cdots + O(\varepsilon^k) + \cdots + O(1) + O(1) = 0.$$

Regardless of the orientation of the small coefficient, terms can be balanced and hence there is always a possibility of having a root of order 1. This is the first possibility stated in Theorem 3.

(ii) $x \sim O(1/\varepsilon^r), r > 0$:
For this case, the orders of magnitudes of all terms read

$$O(1/\varepsilon^{nr}) + O(1/\varepsilon^{(n-1)r}) + \cdots + O(1/\varepsilon^{\mathrm{rm}-k}) + \cdots + O(1) = 0.$$

Balancing of the leading term is impossible unless $m = n$. For this special case, the equation is