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Ruled surfaces of non-degenerate third fundamental forms in Minkowski 3-spaces

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ABSTRACT

In this paper, we study ruled surfaces in a Minkowski 3-space satisfying some equation in terms of a position vector field and Laplacian operator with respect to non-degenerate third fundamental form. Furthermore, we give a new example of null scroll in a Minkowski 3-space.

1. Introduction

Let $\vec{x} : M \to \mathbb{E}^3$ be an isometric immersion of a surface in an Euclidean 3-space. Denote by $\vec{H}$ and Δ, respectively, the mean curvature vector field and the Laplacian operator of M with respect to the Riemannian metric g on M induced from that of $\mathbb{E}^3$. Then, as is well known $\Delta\vec{x} = -2\vec{H}$. Takahashi [9] proved that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta\vec{x} = \lambda\vec{x},\ \lambda \in \mathbb{R}$, and Garay [2] extended it to the hypersurfaces, that is, he studied the hypersurfaces in $\mathbb{E}^{n+1}$ for which

$$\Delta\vec{x} = A\vec{x}, \quad A \in \mathrm{Mat}(n+1, \mathbb{R}), \tag{1.1}$$

where $\mathrm{Mat}(n+1, \mathbb{R})$ is the set of $(n+1) \times (n+1)$-real matrices.

Recently, in [4] Kaimakamis and Papantoniou studied surfaces of revolution with non-degenerate second fundamental form II in a Minkowski 3-space $\mathbb{L}^3$ satisfying the condition

$$\Delta^{II}\vec{x} = A\vec{x}, \quad A \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.2}$$

where Δ^{II} is the Laplacian operator with respect to non-degenerate second fundamental form II.

In this paper, we will investigate ruled surfaces satisfying some equation in terms of a position vector field and Laplacian operator with respect to non-degenerate third fundamental form III of the surface in a Minkowski 3-space $\mathbb{L}^3$, that is

$$\Delta^{III}\vec{x} = A\vec{x}, \quad A \in \mathrm{Mat}(3, \mathbb{R}), \tag{1.3}$$

where Δ^{III} is the Laplacian operator with respect to non-degenerate third fundamental form III.

In a Minkowski 3-space, the space-like ruled minimal surfaces were studied by Kobayashi [7] and the time-like ruled minimal surfaces by Van de Woestijne [10]. In particular, the following theorems were proved.

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Theorem A [7]. *Every space-like ruled minimal surface in a Minkowski 3-space is congruent to part of one the following surfaces:*

1. *a space-like plane;*
2. *the helicoid of the 1st kind;*
3. *the helicoid of the 2nd kind;*
4. *the conjugate of Enneper's surface of the 2nd kind.*

Theorem B [10]. *Every time-like ruled minimal surface in a Minkowski 3-space is congruent to part of one the following surfaces:*

1. *a time-like plane;*
2. *the helicoid of the 1st kind;*
3. *the helicoid of the 2nd kind;*
4. *the helicoid of the 3rd kind;*
5. *the conjugate of Enneper's surface of the 2nd kind;*
6. *a flat B-scroll.*

2. Preliminaries

Let $\mathbb{L}^3$ be a Minkowski 3-space with the scalar product of index 1 given by $\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a standard rectangular coordinate system of $\mathbb{L}^3$. A vector x of $\mathbb{L}^3$ is said to be *space-like* if $\langle x, x\rangle > 0$ or $x = 0$, *time-like* if $\langle x, x\rangle < 0$ and *light-like or null* if $\langle x, x\rangle = 0$ and $x \neq 0$. A time-like or null vector in $\mathbb{L}^3$ is said to be *causal*. A curve in $\mathbb{L}^3$ is called *space-like, time-like or null* if its tangent vector field is space-like, time-like or null, respectively.

We denote a surface M in $\mathbb{L}^3$ by

$$\vec{x}(s,t) = (x_1(s,t), x_2(s,t), x_3(s,t)).$$

Let U be the standard unit normal vector field on a surface M defined by $U = \frac{\vec{x}_s \times \vec{x}_t}{\|\vec{x}_s \times \vec{x}_t\|}$, where $\vec{x}_s = \frac{\partial \vec{x}(s,t)}{\partial s}$. Then the first fundamental form I of a surface M is defined by

$$I = E\,ds^2 + 2F\,ds dt + G\,dt^2,$$

where $E = \langle \vec{x}_s, \vec{x}_s\rangle,\ F = \langle \vec{x}_s, \vec{x}_t\rangle,\ G = \langle \vec{x}_t, \vec{x}_t\rangle$.We define the second fundamental form II and the third fundamental form III of M by, respectively

$$II = e\,ds^2 + 2f\,ds dt + g\,dt^2,$$
$$III = X\,ds^2 + 2Y\,ds dt + Z\,dt^2,$$

where

$$e = \langle \vec{x}_{ss}, U\rangle, \quad f = \langle \vec{x}_{st}, U\rangle, \quad g = \langle \vec{x}_{tt}, U\rangle,$$
$$X = \langle U_s, U_s\rangle, \quad Y = \langle U_s, U_t\rangle, \quad Z = \langle U_t, U_t\rangle.$$

If the third fundamental form III is non-degenerate, then it can be regarded as a (pseudo-) Riemannian metric, and the Laplacian operator Δ^{III} with respect to III can be defined formally on the (pseudo-) Riemannian manifold (M, III). Using classical notation, we define the Laplacian operator Δ^{III} by (cf. [8])

$$\Delta^{III} = -\frac{1}{\sqrt{|\mathcal{T}|}} \sum_{i,j}^{2} \frac{\partial}{\partial x^i}\left(\sqrt{|\mathcal{T}|}\,T^{ij}\frac{\partial}{\partial x^j}\right), \tag{2.1}$$

where $T_{11} = X, T_{12} = Y, T_{22} = Z$, $\mathcal{T} = \det(T_{ij})$ and $(T^{ij}) = (T_{ij})^{-1}$.

Now, we define a ruled surface M in $\mathbb{L}^3$. Let I and J be open intervals containing 0 in the real line $\mathbb{R}$. Let $\alpha = \alpha(s)$ be a curve of J into $\mathbb{L}^3$ and $\beta = \beta(s)$ a vector field along α. Then, a ruled surface M is defined by the parametrization given as follows:

$$\vec{x} = \vec{x}(s,t) = \alpha(s) + t\beta(s), \quad s \in J,\ t \in I.$$

For such a ruled surface, α and β are called the *base curve* and the *director vector field*, respectively. According to the causal character of α' and β, there are four possibilities:

(1) α' and β are non-null and linearly independent.
(2) α' is null and β is non-null with $\langle \alpha', \beta\rangle \neq 0$.
(3) α' is non-null and β is null with $\langle \alpha', \beta\rangle \neq 0$.
(4) α' and β are null with $\langle \alpha', \beta\rangle \neq 0$.

It is easily to see that, with an appropriate change of the curve α, cases (2) and (3) reduce to (1) and (4), respectively (For the details, see [5]).

First of all, we consider the ruled surface of the case (1). In this case, the ruled surface M is said to be *cylindrical* if the director vector field β is constant and *non-cylindrical* otherwise.