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Parameter extension for combined hybrid finite element methods and application to plate bending problems ☆

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ABSTRACT

Based on a weighted average of the modified Hellinger–Reissner principle and its dual, the combined hybrid finite element (CHFE) method was originally proposed with a combination parameter limited in the interval $(0,1)$. In actual computation this parameter plays an important role in adjusting the energy error of discretization models. In this paper, a novel expression of the combined hybrid variational form is used to show the relationship between the resultant method and some Galerkin/least-squares stabilized finite scheme for plate bending problems. The choice of combination parameter is then extended to $(-\infty,0)\bigcup(0,1)$. Existence, uniqueness and convergence of the solution of discrete schemes are proved, and the advantage of the parameter extension in computation is discussed. As an application, improvement of Adini's rectangular element by the CHFE approach is performed.

1. Introduction

The combined hybrid finite element (CHFE) method is a special stabilized mixed method developed in recent years for elasticity problems [1–3]. Based on a weighted average of the formulations of Hellinger–Reissner principle and its dual, the primal hybrid variational principle, this method does not require finite element pairs of stress and displacement spaces to satisfy the inf-sup or LBB conditions, and, for any given combination parameter $\alpha\in(0,1)$, it always yields a convergent numerical solution.

For fourth-order plate bending problems, due to the C^1-continuity requirement, determination of suitable displacement shape functions is much more complex than those needed for C^0-continuity. This C^1 difficulty has resulted in many mixed approaches such as hybrid formulations and least-squares methods which include the use of Lagrangian multiplier and penalty strategies (see the papers [4–18] and the references therein for details). Because of the 'saddle-point' nature of the hybrid methods, the displacement and bending moments approximations are required to satisfy the inf-sup stability condition (see, e.g. [4,6]). In [19] the CHFE approach was extended to the numerical analysis of the plate bending problems to avoid the inf-sup difficulty and to yield stabilized hybrid schemes in the sense that the displacement and bending moments variables are approximated independently.

Due to elimination of the stress/moments parameters at the element level, the CHFE method preserves the convenience of the standard Galerkin displacement scheme. Moreover, this method is shown to be of an energy-error-adjusting mechanism

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[20–22], i.e. for given displacement and stress/moments modes, by changing the combination parameter α in the interval $(0,1)$ one can adjust the energy of the discretization model so as to reduce the energy error. In [20–22], it was shown numerically that the smaller the energy error is, the better the accuracy of the scheme will be. However, in some applications there are circumstances that the energy error of a CHFE scheme with a special displacement approximation can not be reduced for the parameter $\alpha \in (0,1)$ [21], so it is impossible to attain higher numerical accuracy at coarse meshes for the corresponding CHFE scheme by choosing an appropriate α in $(0,1)$. Hence, a further study of the energy-error-adjusting mechanism of the CHFE method seems to be required.

In this paper, a new survey of the CHFE method is carried out for plate bending problems so as to disclose some new interesting aspects of the method. By using a novel equivalent expression, the CHFE scheme is shown to enjoy the form of some Galerkin/least-squares stabilized finite element method [23,24]. This observation then leads to an extension of the combination parameter interval from $(0,1)$ to $(-\infty,0)\bigcup(0,1)$. As an application, improvement of Adini's plate element by the CHFE method is investigated.

Throughout the paper the letter C represents a positive constant which is independent of the mesh size $h = \max_{\Omega_i}\{h_i\}$ and may be different at its each occurrence.

2. Combined hybrid variational principle

Considering the following plate bending problem:

$$\begin{cases} \mathbf{divdiv}\sigma = f & \text{in } \Omega, \\ \sigma = \mathbf{m}(\mathbf{D}_2 u) & \text{in } \Omega, \\ u = \nabla u \cdot \mathbf{n} = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \Re^2$ is a bounded open set, u represents the vertical deflection, $\sigma = (\sigma_{ij})$ $(i,j = 1,2)$ denotes the symmetric bending moment tensor (i.e. $\sigma_{12} = \sigma_{21}$), $\mathbf{divdiv}\sigma = \partial_{11}\sigma_{11} + 2\partial_{12}\sigma_{12} + \partial_{22}\sigma_{22}$ with $\partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$ $(i,j = 1,2)$,

$$\mathbf{D}_2 u = \begin{pmatrix} \partial_{11}u & \partial_{12}u \\ \partial_{12}u & \partial_{22}u \end{pmatrix}, \quad \mathbf{m}(\sigma) = \begin{pmatrix} \sigma_{11} + \nu\sigma_{22} & (1-\nu)\sigma_{12} \\ (1-\nu)\sigma_{12} & \nu\sigma_{11} + \sigma_{22} \end{pmatrix}$$

with $\nu \in (0, 0.5)$ the Poisson's coefficient, and $\mathbf{n}$ is the unit outer normal vector along $\partial\Omega$.

The combined hybrid variational principle corresponding to the problem (2.1) reads as [19]:

$$\inf_{(v,v_c)\in U\times U_c} \sup_{\tau\in\mathbf{V}} \left\{ \frac{1-\alpha}{2} d(v,v) - f(v) - b_1(\tau, v - v_c) + \alpha\left[b_2(\tau,v) - \frac{1}{2}a(\tau,\tau)\right]\right\}, \tag{2.2}$$

where $T_h = \{\Omega_i\}$ denotes a subdivision of Ω, h_i the diameter of Ω_i, and

$$\mathbf{V} = \prod_{\Omega_i\in T_h} H(\mathbf{divdiv}, \Omega_i) = \prod_{\Omega_i\in T_h} \left\{\tau \in \left(L^2(\Omega_i)\right)^4_s; \mathbf{divdiv}\tau \in L^2(\Omega_i)\right\},$$

$$U = \left\{ v \in \prod_{\Omega_i\in T_h} H^2(\Omega_i);\ v = \nabla v \cdot \mathbf{n} = 0,\ on\ \partial\Omega \right\},$$

$$U_c = H_0^2(\Omega)/\prod_{\Omega_i\in T_h} H_0^2(\Omega_i) = \left\{ trace\ of\ \ v \in H_0^2(\Omega)\ on\ boundaries \prod_{\Omega_i\in T_h} \partial\Omega_i \right\}$$

are respectively the symmetric bending moment vector space, the deflection space, and the interelemental boundary deflection space, $\left(L^2(\Omega_i)\right)^4_s$ the space of square integrable 2×2 symmetric tensors, and

$$a(\sigma,\tau) = \int_\Omega \mathbf{m}^{-1}(\sigma) : \tau\, d\mathbf{x},$$

$$b_1(\tau, v - v_c) = \sum \oint_{\partial\Omega_i} [M_{nn}(\tau)\nabla(v - v_c)\cdot\mathbf{n} + M_{ns}(\tau)\nabla(v - v_c)\cdot\mathbf{s} - Q_n(\tau)(v - v_c)]ds,$$

$$b_2(\tau, v) = \sum \int_{\Omega_i} \tau : \mathbf{D}_2 v\, d\mathbf{x},$$

$$d(u, v) = \sum \int_{\Omega_i} \mathbf{m}(\mathbf{D}_2 u) : \mathbf{D}_2 v\, d\mathbf{x},$$

$$f(v) = \int_\Omega f v\, d\mathbf{x},$$

$$M_{nn}(\tau) = (\tau\mathbf{n})\cdot\mathbf{n}, \quad M_{ns}(\tau) = (\tau\mathbf{n})\cdot\mathbf{s}, \quad Q_n(\tau) = \nabla(tr(\tau))\cdot\mathbf{n},$$

$\mathbf{n}$ = unit outer normal vector along $\partial\Omega_i$,

$\mathbf{s}$ = unit tangent vector along $\partial\Omega_i$.