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A priori reduction method for solving the two-dimensional Burgers' equations

C. Allery [a,*], A. Hamdouni [a], D. Ryckelynck [b], N. Verdon [c]

[a] LEPTIAB, Université de La Rochelle, France
[b] Centre des Matériaux, UMR CNRS 7633, Ecole des mines de Paris, France
[c] Laboratoire J.-A. Dieudonné, UMR CNRS 6621, Université de Nice - Sophia Antipolis, France

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ABSTRACT

The two-dimensional Burgers' equations are solved here using the *A Priori Reduction* method. This method is based on an iterative procedure which consists in building a basis for the solution where at each iteration the basis is improved. The method is called *a priori* because it does not need any prior knowledge of the solution, which is not the case if the standard Karhunen–Loéve decomposition is used. The accuracy of the APR method is compared with the standard Newton–Raphson scheme and with results from the literature. The APR basis is also compared with the Karhunen–Loéve basis.

1. Introduction

Burgers' equations are one of the most important non-linear equations. Despite their apparent simplicity, they are similar to the Navier–Stokes equations, without the pressure term, which is why they have been intensively studied in the field of fluid dynamics, where they are considered in particular to be a simplified model of turbulence [1]. Because of their simplicity, they have frequently been studied in many other fields to obtain a better understanding of non-linear phenomena (see for instance the review of Bec and Khanin [2]). In this article we focussed on the two-dimensional Burgers' equation, expressed as:

$$\begin{cases} \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \\ \frac{\partial v}{\partial t} + u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \end{cases} \tag{1}$$

with the following initial conditions:

$$\begin{cases} u(x,y,0) = f_1(x,y), \quad \forall (x,y) \in \Omega, \\ v(x,y,0) = f_2(x,y), \quad \forall (x,y) \in \Omega \end{cases}$$

and the boundary conditions:

$$\begin{cases} u(x,y,t) = g_1(x,y,t), \quad \forall x,y \in \partial\Omega \quad \text{and} \quad t > 0, \\ v(x,y,t) = g_2(x,y,t), \quad \forall x,y \in \partial\Omega \quad \text{and} \quad t > 0, \end{cases}$$

* Corresponding author.
E-mail address: callery@univ-lr.fr (C. Allery).

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where $\Omega = \{(x,y) : a \leqslant x \leqslant b, a \leqslant y \leqslant b\}$ is the computational domain and $\partial\Omega$ its boundary, $u(x,y,t)$ and $v(x,y,t)$ are the velocity components and it is known that the functions f_1, f_2, g_1 and g_2 to ensure that the problem (1) is well-posed. Finally, ν is the diffusion coefficient.

This problem can be solved numerically by linearizing it. However in this case, if there are several branches of solutions, we will not be able to find all of them. If we want to obtain all the solutions, the non-linearized problem must be solved. In this case, the Newton–Raphson method is usually used [3,4], but this method is very computer-time consuming. In order to reduce the time the A Priori Reduction (APR) method was applied. This approach consists in projecting the discretized problem onto a basis we defined in order to obtain a reduced model that can be quickly and accurately solved with standard numerical schemes such as Newton–Raphson. The originality of the method lies in the way the basis is constructed. At each iteration of computation, the basis is adapted: first the old basis is improved using a Karhunen–Loéve decomposition while in the second phase the improved basis is expanded with Krylov vectors [5]. This method, originally introduced by Ryckelynck for solving non-linear thermo-mechanical problems [6–8], can also be used to find a basis of the solution to build a low order dynamical system [9,10]. In this article, the algorithm was slightly modified in order to further decrease the computing time. The following section presents the APR algorithm. The results obtained for the two-dimensional Burgers' equation were then compared to those calculated with the Newton–Raphson method. Finally, the APR basis was compared to the Karhunen–Loéve decomposition (also known as the POD (Proper Orthogonal Decomposition) in the field of fluid mechanics), which is the most popular method for building a physical basis.

2. The method of A Priori Reduction

2.1. APR basis

The APR is an iterative method whose aim is to find a good representative basis $\boldsymbol{\varphi}$ of the unknown quantity of the equation and to express it as a decomposition in this basis:

$$\mathbf{U}(\mathbf{x},t) = \sum_{i=1}^{N} a_i(t)\boldsymbol{\varphi}^i(\mathbf{x}). \tag{2}$$

Unlike the POD, which is an a *posteriori* technique, we do not have any information about the solution to the problem. To build the basis, iterative improvements of an initial basis are used.The steps of the APR method are the following:

(i) Initialization of the basis $\boldsymbol{\varphi}^1$
In preliminary studies (see [11] for more details) it was shown that the choice of the initial basis is not crucial because the method is able to find the correct basis and so the correct solution from any initial guess concerning the basis. For all the computations presented here, the usual choice for the initial basis was the discretized initial condition.

(ii) Projection of the full discretized problem into the APR basis $\boldsymbol{\varphi}$
The 2D Burgers' equations (1) can be discretized using a finite volume scheme, finite element scheme or finite-difference scheme. The discretized set of equations obtained can now be expressed as:

$$\frac{d\mathbf{U}_h}{dt} + \mathcal{F}_h(\mathbf{U}_h) = \mathcal{G}_h, \tag{3}$$

where $\mathbf{U}_h = {}^t(u_h, v_h)$ is the vector of the discretized velocities, $\mathcal{F}_h$ represents the discretized non-linear operator of the two-dimensional Burgers' equation while $\mathcal{G}_h$ is the discretized second member.
Let us now consider that we are on the kth iteration of the APR process. So we have

$$\mathbf{U}_h(t) = \sum_{i=1}^{N} a_i(t)\boldsymbol{\varphi}_h^i, \tag{4}$$

where N (very small $\simeq 10$) is the number of APR vectors bases at the kth iteration. By introducing this decomposition into the discretized Eq. (3) and by projecting it onto the basis φ a low order model of size N is obtained:

$${}^t\boldsymbol{\varphi}_h^j \frac{d\mathbf{U}_h}{dt} + {}^t\boldsymbol{\varphi}_h^j \mathcal{F}_h\left(\sum_{i=1}^{N} a_i(t)\boldsymbol{\varphi}_h^i\right) = {}^t\boldsymbol{\varphi}_h^j \mathcal{G}_h \quad \text{for } 1 \leqslant j \leqslant N. \tag{5}$$

Solving this set of equations is faster than solving problem (3).

(iii) Solving the reduced problem (5) for the whole time interval T considered
Solving this problem provides knowledge of the temporal coefficients $a_i(t_j)$ for $t_j \in T$. Then by using (4) we obtain $\mathbf{U}_h(t)$ and compute for each time step t_j the residuals $\mathbf{R}(t_j)$ associated with (3):

$$\mathbf{R}(t_j) = \frac{d\mathbf{U}_h}{dt} + \mathcal{F}_h(\mathbf{U}_h) - \mathcal{G}_h.$$