Applied Mathematics and Computation 217 (2010) 1266–1276

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On conjugate points and the Leitmann equivalent problem approach

F.O.O. Wagener

CeNDEF, Department of Quantitative Economcis, Universiteit van Amsterdam, Roetersstraat 11, 1018 WB, Amsterdam, The Netherlands

ARTICLE INFO

Keywords:
Calculus of variations
Equivalent problems
Conjugate points
Rectifying coordinates
Sufficient conditions

ABSTRACT

This article extends the Leitmann equivalence method to a class of problems featuring conjugate points. The class is characterised by the requirement that the set of indifference points of a given problem forms a finite stratification.

1. Introduction

In this article, the Leitmann equivalence method [1–3] that gives absolute extrema of calculus of variations problems is extended to a class of problems that feature conjugate points.

Recall that the Leitmann equivalence method consists in considering a classical field of extremals as a coordinate transformation: the extremals of the transformed problem are then constants. Optimality of the transformed extremals is obtained by using Carathéodory's equivalent problem approach [4]. This gives the sufficiency theory of the classical Calculus of Variations a particularly simple and elegant form.

However, the above summary of the equivalence method also indicates one of its main limitations: the method breaks down when extremals intersect, that is, when the field of extremals fails to define an invertible coordinate transformation. It may fail globally, when a point in the extended state space is reached by several extremals, or locally, when the differential of the transformation at a point fails to have full rank. The latter points are classically called conjugate.

The present article is a step in the direction of removing this limitation. First, only the optimal part of the extremals of the field are considered. The obvious but important thing to note is that an extremal should be restricted to its maximal domain of optimality, which is defined as the maximal interval of definition such that no other extremal in the field having the same endpoint gives a lower value to the objective functional. If the domain of optimality does not coincide with the integration domain of the objective functional, several situations may arise: either the extremal cannot be extended to a larger domain of definition or it fails to be optimal on a larger domain. In the latter situation, the endpoint is either a conjugate point, an indifference point, meaning that it is reached by several extremals all giving the objective functional the same value, or it may be an infimal point: in this case, the point is reached by an infinity of extremals, but the set of associated values of the objective functional has no minimal value.

Attention is restricted to the situation that there are no infimal points, and that the set of indifference points forms a finite stratification; this means that it is the union of finitely many open differential manifolds, possibly of different dimensions, such that each manifold that intersects the closure of another manifold is actually contained in this closure. Having specified in this way the structure of the set of extremals restricted to their domains of optimality, the second step is to show that an element of this set actually minimises the objective functional also on the much larger set of all admissible trajectories. In the proof of this second step, it is sufficient to consider the generic situation of a smooth non-extremal trajectory attaining a lower value of the objective functional than all extremals of the field, and intersecting only finitely many indifference

E-mail address: f.o.o.wagener@uva.nl

0096-3003/$ - see front matter
doi:10.1016/j.amc.2010.02.036

manifolds finitely often. The main technical point then is to show how these intersections can be removed without changing the value of the objective functional too much, ending up with a trajectory that has no intersections with any indifference manifold, but still realises a lower value of the objective functional than any extremal. Leitmann rectification now immediately shows the impossibility of this situation. The theorem is illustrated by its application to a relatively simple problem that features indifference points.

Leitmann's rectifying coordinates are closely related to Kneser's normal coordinates of a field. Kneser [5], Section 22, considered parametric problems having a positive integrand. He also used a field of extremals as a coordinate transformation; however, as the second normal variable he took the accumulated value of the objective functional along the extremal. This ensures that the integrand of the objective functional is constant along extremals and it eliminates the need to consider an equivalent problem. In Section 22.IV Kneser demonstrates that in normal coordinates, the resulting variational problem can be solved by inspection. The restriction to problems with positive integrands was forced by the need to have the integrand transform to a simple form, as the method of equivalent problems was not known at the time. The central idea of the present article, to replace segments of non-extremal trajectories by segments of extremal trajectories, has its roots in the so-called Calculus of Variations in the Large, where, however, it is applied to several relatively short segments [see 4, Section 385].

There is a rapidly growing literature on the subject of Leitmann rectification, showing that the method is general and in principle applicable to all kinds of problems connected to the Calculus of Variations; see [2,3,6–12].

2. The problem

2.1. Preliminary definitions

In the following, any $\mathscr{C}^k$ function defined on a closed set G is always assumed to be the restriction of a $\mathscr{C}^k$ function defined on an open neighbourhood of G.

Let points $a, b \in \mathbb{R}$ be given, $a < b$, as well as points $\alpha, \beta \in \mathbb{R}^n$. The set $\mathscr{X} = [a, b] \times \mathbb{R}^n$ is called the *extended state space*. Let $L : \mathscr{X} \times \mathbb{R}^n \to \mathbb{R}$ be a $\mathscr{C}^2$ function on the *extended tangent space* $\mathscr{T} = \mathscr{X} \times \mathbb{R}^n$. Writing $L = L(t, x, v)$ with $(t, x) \in \mathscr{X}$ and $v \in \mathbb{R}^n$, it is assumed that

$$L_{vv}(t, x, v) > 0 \quad \text{for all } (t, x) \in \mathscr{X}.$$

Finally, for $a < T \leqslant b$, let $\mathscr{A}_T$ be the space of absolutely continuous functions $x : [a, T] \to \mathbb{R}^n$ that are such that $x(a) = \alpha$; an element of $\mathscr{A}_T$ will be called a *trajectory (starting in α)* in the following. Let moreover $\mathscr{B}_{T,X}$ be the subset of $\mathscr{A}_T$ of trajectories x that satisfy $x(T) = X$.

The space $\mathscr{A}_T$ will be equipped with the topology induced by the metric

$$d_T(x_1, x_2) = \int_a^T |\dot{x}_1(t) - \dot{x}_2(t)| \mathrm{d}t.$$

Recall that the set of $\mathscr{C}^\infty$ trajectories is dense in $\mathscr{A}_T$ with respect to this metric.

2.2. The minimisation problem

Introduce for $a < T \leqslant b$ the functional $J_T : \mathscr{A}_T \to \mathbb{R}$ that is defined by

$$J_T(x) = \int_a^T L(t, x, \dot{x}) \mathrm{d}t.$$

In this article, I consider the standard problem to find a minimiser of J_b on $\mathscr{B}_{b,\beta}$, that is, an element $x_0 \in \mathscr{A}_b$ such that $x_0(b) = \beta$ and such that $J_b(x_0) \leqslant J_b(x)$ for all $x \in \mathscr{B}_{b,\beta}$.

Recall from the Calculus of Variations that if $x \in \mathscr{B}_{b,\beta}$ minimises J_b, then it is necessarily a solution to the Euler–Lagrange equation

$$L_x(t, x, \dot{x}) - \frac{\mathrm{d}}{\mathrm{d}t} L_v(t, x, \dot{x}) = 0, \tag{1}$$

satisfying the boundary conditions

$$x(a) = \alpha, \quad x(b) = \beta.$$

In general, solutions to (1) are called *extremals*. The regularity assumption $L_{vv} > 0$ implies that every extremal is at least $\mathscr{C}^2$. Introduce the subspaces $\mathscr{E}_T \subset \mathscr{A}_T$ and $\mathscr{E}_{T,X} \subset \mathscr{B}_{T,X}$ of trajectories in $\mathscr{A}_T$ and $\mathscr{B}_{T,X}$, respectively, that are extremals.