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Heat problems for a starlike shaped plate

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ABSTRACT

The Fourier solution of the heat problem for a circular plate is generalized to a starlike shaped plate. We show that the classical solution can be used even in this more general case, provided that a suitable change of variables in the polar co-ordinate system is performed.

1. Introduction

One of the well known applications of the Bessel functions [1] is related to the separation of variables in the partial differential equation representing the heat equation for a circular plate, namely, denoting by B a circular domain of radius $r = 1$ centered at the origin, by ∂B the relevant boundary, by κ a constant representing the known diffusivity, and by $f(x,y) \in C^0(B)$ the initial temperature, the solution $u(x,y,t) \in [C^2(\mathring{B}) \times C^1(\mathbf{R}^+)] \cap C^0[\overline{B} \times \mathbf{R}^+]$ of the differential problem

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), & \text{in } \mathring{B}, \\ u(x,y,t)|_{(x,y)\in\partial B} = 0, & u(x,y,0) = f(x,y) \end{cases} \tag{1.1}$$

putting

$$U(\rho,\theta,t) = u(\rho\cos\theta, \rho\sin\theta, t), \quad F(\rho,\theta) = f(\rho\cos\theta,\rho\sin\theta), \tag{1.2}$$

can be represented by the Fourier expansion in terms of exponential, circular and Bessel functions

$$u(x,y,t) = U(\rho,\theta,t) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}(A_{m,k}\cos m\theta + B_{m,k}\sin m\theta)J_m\left(j_k^{(m)}\rho\right)\exp\left[-\left(j_k^{(m)}\right)^2\kappa t\right], \tag{1.3}$$

where the coefficients $A_{m,k}, B_{m,k}$ are given by

$$\begin{cases} A_{0,k} = \frac{1}{\pi\left[J_1(j_k^{(0)})\right]^2}\int_0^1 \zeta\left[\int_0^{2\pi}F(\zeta,\tau)\,d\tau\right]J_0\left(j_k^{(0)}\zeta\right)d\zeta, \\ A_{m,k} = \frac{2}{\pi\left[J_{m+1}(j_k^{(m)})\right]^2}\int_0^1 \zeta\left[\int_0^{2\pi}F(\zeta,\tau)\cos m\tau\,d\tau\right]J_m\left(j_k^{(m)}\zeta\right)d\zeta, \\ B_{m,k} = \frac{2}{\pi\left[J_{m+1}(j_k^{(m)})\right]^2}\int_0^1 \zeta\left[\int_0^{2\pi}F(\zeta,\tau)\sin m\tau\,d\tau\right]J_m\left(j_k^{(m)}\zeta\right)d\zeta, \end{cases} \tag{1.4}$$

and $j_k^{(m)}$ denote the zeros of the Bessel function J_m.

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The heat problem for a plate with a general shape is often reduced to the circular case by using the conformal mappings technique (see e.g. [2,3]), however only very special cases can be treated analytically by using this method, since only few explicit equations for the relevant conformal mappings are known.

We consider in this paper an extension of the classical two-dimensional theory to the case of a starlike domain, i.e. a domain $\mathscr{D}$, which is normal with respect to the polar co-ordinate system.

$\partial\mathscr{D}$ can be interpreted as an *anisotropically stretched unit circle*.

We introduce in the x, y plane the ordinary polar co-ordinates:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta, \tag{1.5}$$

and the polar equation of $\partial\mathscr{D}$

$$\rho = r(\theta) \quad (0 \leqslant \theta \leqslant 2\pi), \tag{1.6}$$

where $r(\theta) \in C^2[0, 2\pi]$. We suppose the domain $\mathscr{D}$ satisfies

$$0 < A \leqslant \rho \leqslant r(\theta)$$

and therefore $\min_{\theta\in[0,2\pi]} r(\theta) > 0$.

We introduce the stretched radius ρ^* such that

$$\rho = \rho^* r(\theta), \tag{1.7}$$

and the curvilinear (i.e. stretched) co-ordinates ρ^*, θ, in the plane x, y,

$$x = \rho^* r(\theta)\cos\theta, \quad y = \rho^* r(\theta)\sin\theta. \tag{1.8}$$

Therefore, $\mathscr{D}$ is obtained assuming $0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant \rho^* \leqslant 1$.

We show how to modify some classical formulas, and we derive methods to compute the coefficients of Fourier-type expansions representing solutions of some classical problems. We have considered only regular functions for the boundary and the boundary data, but the relevant results can be easily generalized considering weakened hypotheses (see Remark 2, below).

2. The Laplacian in stretched polar co-ordinates

We consider a $C^2(\mathring{\mathscr{D}})$ function $u(x,y) = u(\rho\cos\theta, \rho\sin\theta) = U(\rho,\theta)$ and the Laplace operator in polar co-ordinates

$$\Delta_2 u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2}. \tag{2.1}$$

We start representing this operator in the new stretched co-ordinate system ρ^*, θ.

Putting

$$V(\rho^*, \theta) = u[\rho^* r(\theta)\cos\theta, \rho^* r(\theta)\sin\theta] = U(\rho, \theta),$$

we find

$$\frac{\partial U}{\partial \rho} = \frac{1}{r(\theta)}\frac{\partial V}{\partial \rho^*}, \tag{2.2}$$

$$\frac{\partial^2 U}{\partial \rho^2} = \frac{1}{r^2(\theta)}\frac{\partial^2 V}{\partial \rho^{*2}}, \tag{2.3}$$

$$\frac{\partial U}{\partial \theta} = -\rho^*\frac{r'(\theta)}{r(\theta)}\frac{\partial V}{\partial \rho^*} + \frac{\partial V}{\partial \theta}, \tag{2.4}$$

$$\frac{\partial^2 U}{\partial \theta^2} = \rho^*\frac{2r'^2(\theta) - r(\theta)r''(\theta)}{r^2(\theta)}\frac{\partial V}{\partial \rho^*} + \rho^{*2}\frac{r'^2(\theta)}{r^2(\theta)}\frac{\partial^2 V}{\partial \rho^{*2}} - 2\rho^*\frac{r'(\theta)}{r(\theta)}\frac{\partial^2 V}{\partial \rho^*\partial \theta} + \frac{\partial^2 V}{\partial \theta^2}. \tag{2.5}$$

Substituting we find our result, i.e.

$$\begin{aligned}\Delta_2 u &= \frac{\partial^2 U}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial U}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial \theta^2}\\ &= \frac{1}{r^2(\theta)}\left[1 + \frac{r'^2(\theta)}{r^2(\theta)}\right]\frac{\partial^2 V}{\partial \rho^{*2}} + \frac{1}{\rho^* r^2(\theta)}\left[1 + \frac{2r'^2(\theta) - r(\theta)r''(\theta)}{r^2(\theta)}\right]\frac{\partial V}{\partial \rho^*} - 2\frac{r'(\theta)}{\rho^* r^3(\theta)}\frac{\partial^2 V}{\partial \rho^*\partial \theta} + \frac{1}{\rho^{*2} r^2(\theta)}\frac{\partial^2 V}{\partial \theta^2}.\end{aligned} \tag{2.6}$$

For $\rho^* = \rho,\ r(\theta) \equiv 1$, we recover the Laplacian in polar co-ordinates.

2.1. An equivalent formulation

For further computations, it is more easy to change the polar equation of $\partial\mathscr{D}$ putting