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Catalan matrix and related combinatorial identities

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ABSTRACT

We introduce the notion of the Catalan matrix $\mathscr{C}_n[x]$ whose non-zero elements are expressions which contain the Catalan numbers arranged into a lower triangular Toeplitz matrix. Inverse of the Catalan matrix is derived. Correlations between the matrix $\mathscr{C}_n[x]$ and the generalized Pascal matrix are considered. Some combinatorial identities involving Catalan numbers, binomial coefficients and the generalized hypergeometric function are derived using these correlations. Moreover, an additional explicit representation of the Catalan number, as well as an explicit representation of the sum of the first m Catalan numbers are given.

1. Introduction and preliminaries

A number of investigations have been concerned with Toeplitz matrices which possess numbers of various types arranged on the main diagonal and below.

From the historical point of view, the oldest are known as **Pascal matrices**. Various types of Pascal matrices are investigated in [1–3,10]. The generalized Pascal matrix, denoted by $\mathscr{P}_n[x] = [p_{i,j}[x]]$ $(i,j = 1,\dots,n)$ possesses elements of the form

$$p_{i,j}[x] = \begin{cases} x^{i-j}\binom{i-1}{j-1}, & i-j \geqslant 0 \\ 0, & i-j < 0. \end{cases} \tag{1.1}$$

In the case $x = 1$ the generalized Pascal matrix reduces to Pascal matrix, denoted by $\mathscr{P}_n = [p_{i,j}]$ $(i,j = 1,\dots,n)$.

The lower triangular matrices generated from the Fibonacci numbers are known as **Fibonacci matrices**. The $n \times n$ Fibonacci matrix $\mathscr{F}_n = [f_{i,j}]$ $(i,j = 1,\dots,n)$ is defined by

$$f_{i,j} = \begin{cases} F_{i-j+1}, & i-j+1 \geqslant 0 \\ 0, & i-j+1 < 0, \end{cases} \tag{1.2}$$

where F_k is the kth Fibonacci number [5]. The inverse and Cholesky factorization of the Fibonacci matrix are given in [5]. In [6,12], the authors studied relations between various Pascal matrices and the Fibonacci matrix.

The lower triangular matrices generated by arranging the Stirling numbers of the first kind $s(n,k)$ and of the second kind $S(n,k)$ are studied in [4]. These matrices are known as **Stirling matrices** of the first kind and of the second kind, respectively.

The notions of the **Bernoulli matrix** $\mathscr{B}$ and the **generalized Bernouli polynomial matrix** $\mathscr{B}^{(\alpha)}(x)$ are introduced and investigated in [11].

As an analogy of the Fibonacci matrix, the $n \times n$ **Lucas matrix** $\mathscr{L}_n = [l_{i,j}]$ $(i,j = 1,\dots,n)$ is defined in [13] as

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$$l_{i,j} = \begin{cases} L_{i-j+1}, & i-j \geqslant 0 \\ 0, & i-j < 0, \end{cases} \tag{1.3}$$

where L_k is the kth Lucas number.

In the paper [8] the author investigated the inverse and Cholesky factorization of the matrix $\mathcal{U}_n$ with non-zero entries U_n stored in the lower triangular matrix and defined by the non-degenerated second order sequence $U_{n+1} = AU_n + BU_{n-1}$, where $\delta = \sqrt{A^2 + 4B}$ is real, A, B, U_1 are integers, $U_0 = 0$ (i.e. $A = B$). The author also generalized these results to r-order recurrent sequence satisfying $U_0 = U_{-1} = \cdots = U_{2-r} = 0, U_1$ arbitrary. Results obtained in [8] include known facts about the Fibonacci matrix [5,6], which can be obtained as a special case $U_1 = 1, A = B = 1$.

For the sake of completeness, we restate main definitions from the theory of special functions. The generalized hypergeometric function ${}_pF_q$ is defined as the infinite sum

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \lambda) = \sum_{l=0}^{\infty} \frac{(a_1)_l \cdots (a_p)_l}{(b_1)_l \cdots (b_q)_l} \cdot \frac{\lambda^l}{l!}, \tag{1.4}$$

where

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{1.5}$$

is the well-known Pochhammer function (also known as the rising factorial notation), and $\Gamma(n)$ is the Euler gamma function.

The Catalan numbers $\{C_n\}_{n=0}^{\infty}$ have been widely encountered and investigated. They are the terms of the sequence 1, 1, 2, 5, 14, 42, 132,... where the initial term of the array is $C_0 = 1$, and the nth number is given explicitly in terms of binomial coefficients by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}, \quad n \geqslant 0. \tag{1.6}$$

We also pay attention to the following representation of the Catalan number (see, for example [9]):

$$C_n = \frac{4^n}{\sqrt{\pi}} \cdot \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(2+n)}, \quad n \text{ arbitrary}. \tag{1.7}$$

The following convolution formula involving the Catalan numbers is known

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}, \quad n \geqslant 0, \tag{1.8}$$

as well as the following recurrence relation between two successive Catalan numbers

$$C_{n+1} = \frac{2(2n+1)}{n+2} C_n. \tag{1.9}$$

The main idea in the present paper is usage of the Catalan numbers as the non-zero entries into a lower triangular Toeplitz matrix. We use the notion **Catalan matrix** to denote this matrix.

The main results of the paper are as follows:

1. Explicit representation of the Catalan inverse matrix.
2. Representation for the sum of the first m Catalan numbers and the corresponding recurrent relation.
3. Alternative representation of the Catalan number in terms of the sum of binomial coefficients and the generalized hypergeometric function.
4. Various combinatorial identities and recurrence relations involving Catalan numbers, binomial coefficients and special functions. These results are a continuation of the general ideas used in [6,8,12,13].

The concept of the Catalan matrix is introduced in the second section. The inverse of the Catalan matrix as well as an explicit representation of the sum of first m Catalan numbers are derived. Several correlations between the Catalan matrix $\mathcal{C}_n[x]$ and the generalized Pascal matrix are considered using the inverse of the Catalan matrix. Some combinatorial identities involving Catalan numbers, binomial coefficients, the generalized hypergeometric function and the Pochhammer function are derived. Combinatorial identities arising from the multiplication with the identity row and column vector are obtained in the fourth section.

2. Definitions and auxiliary results

We introduce a kind of lower triangular Toeplitz matrices by arranging Catalan numbers on the main diagonal and below.