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Weighted composition operators from the weighted Bergman space to the weighted Hardy space on the unit ball

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ABSTRACT

We investigate the weighted composition operator from the weighted Bergman space into the weighted Hardy space on the unit ball. As a consequence of the investigation, we also give a characterization for the boundedness and compactness of the operator whose the target space is the Hardy space.

1. Introduction

Let $\mathbb{B}$ denote the open unit ball of the complex n-dimensional Euclidean space $\mathbb{C}^n$ and $H(\mathbb{B})$ denote the space of all holomorphic functions on $\mathbb{B}$. For $0 < p < \infty$ and $\beta \geqslant 0$ we define the *weighted Hardy space* $H^p_\beta(\mathbb{B})$ as follows:

$$H^p_\beta(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \sup_{0<r<1} (1-r)^\beta \int_{\partial\mathbb{B}} |f(r\zeta)|^p \, d\sigma(\zeta) < \infty \right\},$$

where $d\sigma$ is the normalized Lebesgue measure on the boundary $\partial\mathbb{B}$ of $\mathbb{B}$ (see, also [15], as well as [10], for an equivalent definition of the space). We also define the norm $\|\cdot\|_{H^p_\beta}$ on this space as follows:

$$\|f\|^p_{H^p_\beta} = \sup_{0<r<1} (1-r)^\beta \int_{\partial\mathbb{B}} |f(r\zeta)|^p \, d\sigma(\zeta).$$

Furthermore we consider the *weighted Bergman space* $A^p_\alpha(\mathbb{B})(0 < p < \infty, \ \alpha > -1)$ and the *Hardy space* $H^p(\mathbb{B})(0 < p < \infty)$ defined by

$$A^p_\alpha(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}} |f(z)|^p c_\alpha (1-|z|^2)^\alpha dV(z) < \infty \right\},$$

$$H^p(\mathbb{B}) = \left\{ f \in H(\mathbb{B}) : \|f\|^p_{H^p} = \sup_{0<r<1} \int_{\partial\mathbb{B}} |f(r\zeta)|^p d\sigma(\zeta) < \infty \right\},$$

where dV is the normalized Lebesgue measure on $\mathbb{B}$ and c_α is a normalization constant, that is, it is chosen such that $\|1\|_{A^p_\alpha} = 1$. For the sake of convenience, we use the notation $dV_\alpha(z) = c_\alpha(1-|z|^2)^\alpha dV(z)$. By these definitions of spaces, we see that $H^p_\beta(\mathbb{B})$ coincides with $H^p(\mathbb{B})$ when $\beta = 0$ and $H^p_\alpha(\mathbb{B})$ is a closed subspace in $A^p_\alpha(\mathbb{B})$ when $\alpha \geqslant 0$.

Let φ be a holomorphic self-map of $\mathbb{B}$ and $u \in H(\mathbb{B})$. Then φ and u induce a *weighted composition operator* uC_φ on $H(\mathbb{B})$ which is defined by $uC_\varphi f = u(f \circ \varphi)$. This type of operators acting between various spaces of holomorphic functions, has been studied by many authors. For some classical results, see, for example, [3]. When one of the spaces is the Bloch-type or the

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weighted-type space, this operator have been considered, for example, in [2,5–9,16–19,21,26,29–31] (see also the references therein). In the following papers [11,23–25,27], uC_φ acting between the weighted Bergman space and the Hardy space on the unit ball has been studied (see also [4] for the one-dimensional case). These papers characterized the boundedness and compactness of uC_φ in terms of the Carleson measure condition and the behavior of some integral transform which involves symbols u and φ. However they did not give a characterization for the case $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^q(\mathbb{B})$.

In this paper, we will consider the boundedness and compactness of the operator $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^q_\beta(\mathbb{B})$. We also give an estimate for the essential norm of the operator (for some related results, see, for example, [3,19,20,25,28] and the references therein). As a consequence of our main results, we characterize the boundedness and compactness of $uC_\varphi : A^p_\alpha(\mathbb{B}) \to H^q(\mathbb{B})$.

2. Auxiliary results

In order to prove our results, we will need some notation and lemmas.

Lemma 1. *Let $0 < p < \infty$ and $\alpha > -1$. Suppose $f \in H(\mathbb{B})$ and $z \in \mathbb{B}$. Then*

$$|f(z)| \leqslant (1 - |z|^2)^{-\frac{\alpha+n+1}{p}} \|f\|_{A^p_\alpha}.$$

Proof. *This result is well-known. Its proof can be found in [1, Corollary 3.5].* □

Let $\varphi_z (z \in \mathbb{B})$ be the biholomorphic involution of $\mathbb{B}$ described in [13, p. 25]. For $z \in \mathbb{B}$ and $0 < r < 1$, we set $E(z,r) = \varphi_z(r\mathbb{B})$. According to [13, p. 29, Section 2.2.7], $E(z,r)$ consists of all $w \in \mathbb{B}$ that satisfy

$$\frac{|P_z w - c|^2}{(r\rho)^2} + \frac{|w - P_z w|^2}{r^2\rho} < 1,$$

where $P_z w = \frac{\langle w,z\rangle}{\langle z,z\rangle} z$, $c = \frac{(1-r^2)z}{1-(r|z|)^2}$ and $\rho = \frac{1-|z|^2}{1-(r|z|)^2}$.

Lemma 2. *Let $0 < p < \infty$ and $f \in H(\mathbb{B})$. Then*

$$|f(z)|^p \leqslant \frac{(1-r^2)^n}{r^{2n}} \int_{E(z,r)} |f(w)|^p (1 - |w|^2)^{-n-1} dV(w),$$

for each $r, 0 < r < 1$.

Proof. The $\mathscr{M}$-subharmonicity of $|f|^p$ verifies the above estimate. For the detail of the proof, we can refer [22, p. 33]. □

Recall that every $f \in H(\mathbb{B})$ has the homogeneous expansion

$$f(z) = \sum_{k=0}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma,$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multi-index, $|\gamma| = \gamma_1 + \cdots + \gamma_n$ and $z^\gamma = z_1^{\gamma_1} \ldots z_n^{\gamma_n}$. For the homogeneous expansion of f and any integer $j \geqslant 1$, let

$$R_j f(z) = \sum_{k=j}^{\infty} \sum_{|\gamma|=k} c(\gamma) z^\gamma,$$

and $K_j = I - R_j$ where $If = f$ is the identity operator.

Lemma 3. *Let $\alpha > -1$. If $1 < p < \infty$, then $\|R_j f\|_{A^p_\alpha} \to 0$ as $j \to \infty$ for each $f \in A^p_\alpha(\mathbb{B})$.*

Proof. See [25, Corollary 2.1]. □

Lemma 3 and the uniform boundedness principle show that $(R_j)_{j\in\mathbb{N}}$ is a uniformly bounded sequence in $A^p_\alpha(\mathbb{B})$.

Lemma 4. *If uC_φ is bounded from $A^p_\alpha(\mathbb{B})$ into $H^q_\beta(\mathbb{B})$, then*

$$\|uC_\varphi\|_{e,A^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})} \leqslant \liminf_{j\to\infty} \|uC_\varphi R_j\|_{A^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})},$$

where $\|\cdot\|_{e,A^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ and $\|\cdot\|_{A^p_\alpha(\mathbb{B})\to H^q_\beta(\mathbb{B})}$ denote the essential norm and the operator norm respectively.

Proof. See [3, p. 134, Lemma 3.16] □

Lemma 5. *Let $1 < p < \infty$ and $\alpha > -1$. For each $w \in \mathbb{B}$, positive integer j and $f \in A^p_\alpha(\mathbb{B})$,*

$$|R_j f(w)| \leqslant \|f\|_{A^p_\alpha} \sum_{k=j}^{\infty} \frac{\Gamma(k+n+1+\alpha)}{k!\Gamma(n+1+\alpha)} |w|^k$$