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A new hybrid iterative algorithm for variational inequalities ☆

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ABSTRACT

Let H be a real Hilbert space. Let $F : H \to H$ be a strongly monotone and Lipschitzian mapping. Let $\{T_n\}_{n=1}^{\infty} : H \to H$ be an infinite family of non-expansive mappings with common fixed points set $\bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. We devise an iterative algorithm

$$\begin{cases} y_n = x_n - \lambda_n F(x_n), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n W_n y_n, \quad n \geqslant 0, \end{cases}$$

where $\{\lambda_n\}$ is a sequence in $(0,\infty)$, $\{\alpha_n\}$ is a sequence in (0,1) and W_n is the W-mapping. We prove that the sequence $\{x_n\}$ converges in norm to $x^* \in \bigcap_{n=1}^{\infty} Fix(T_n)$ which is the unique solution of the following variational inequality

$$\langle Fx^*, x - x^* \rangle \geqslant 0, \quad \forall x \in \bigcap_{n=1}^{\infty} Fix(T_n).$$

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let $F : C \to C$ be a nonlinear operator. The variational inequality problem is to find $x^* \in C$ such that

$$\mathrm{VI(F, C)} : \quad \langle Fx^*, v - x^* \rangle \geqslant 0, \quad \forall v \in C.$$

Variational inequalities were introduced and studied by Stampacchia [1] in 1964. It is now well-known that a wide classes of problems arising in various branches of pure and applied sciences can be studied in the general and unified framework of variational inequalities. Several numerical methods including the projection and its variant forms, Wiener–Hofp equations, auxiliary principle and descent-type have been developed for solving the variational inequalities and related optimization problems, see [1–10,14–21].

It is known that, if F is a strongly monotone and Lipschitzian mapping on C, then the VI (F, C) has a unique solution. An equally important problem is how to find an approximate solution of the VI(F, C) if any. A great deal of effort has gone into this problem; see [2,6].

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It is also known that the VI(F, C) is equivalent to the fixed point equation

$$x^* = P_C[x^* - \mu F(x^*)], \tag{1.1}$$

where P_C is the projection of H onto the closed convex set C and $\mu > 0$ is an arbitrarily fixed constant. So, fixed point methods can be implemented to find a solution of the $VI(F, C)$ provided F satisfies some conditions and $\mu > 0$ is chosen appropriately. The fixed point formulation (1.1) involves the projection P_C, which may not be easy to compute, due to the complexity of the convex set C. In order to reduce the complexity probably caused by the projection P_C, Yamada [7] (see also [8]) recently introduced a hybrid steepest-descent method for solving the $VI(F, C)$.

Assume that C is the fixed points set of a non-expansive mapping $T : H \to H$; that is, $C = \{x \in H : Tx = x\}$. Recall that T is non-expansive, if $\|Tx - Ty\| \leqslant \|x - y\|$, for $x, y \in H$ and let $Fix(T)$ denote the fixed points set of T. Assume that F is η-strongly monotone and k-Lipschitzian on C. Take a fixed number $\mu \in (0, 2\eta/k^2)$ and a sequence $\{\lambda_n\}$ of real numbers in (0,1) satisfying the conditions below:

(C1) $\lim_{n\to\infty}\lambda_n = 0$,
(C2) $\sum_{n=0}^{\infty}\lambda_n = \infty$,
(C3) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1}^2 = 0$.

Starting with an arbitrary initial guess $x_0 \in H$, one can generate a sequence $\{x_n\}$ by the following algorithm:

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu F(Tx_n), \quad n \geqslant 0. \tag{1.2}$$

Then, Yamada [7] proved that $\{x_n\}$ converges strongly to the unique solution of the $VI(F, C)$. An example of sequence $\{\lambda_n\}$ which satisfies conditions (C1)–(C3) is given by

$$\lambda_n = 1/n^{\sigma}, \quad \text{where } 0 < \sigma < 1.$$

We note that condition (C3) was first used by Lions [9]. It was observed that Lions's conditions on the sequence $\{\lambda_n\}$ excluded the canonical choice $\lambda_n = 1/n$. This was overcome in 2003 by Xu and Kim [10], they proved the strong converges of $\{x_n\}$ to the unique solution of the $VI(F, C)$ if $\{\lambda_n\}$ satisfies conditions (C1), (C2) and (C4)

$$(C4): \quad \lim_{n\to\infty}\lambda_n/\lambda_{n+1} = 1, \quad \text{or equivalently,} \quad \lim_{n\to\infty}(\lambda_n - \lambda_{n+1})/\lambda_{n+1} = 0.$$

It is clear that condition (C4) is strictly weaker than condition (C3), coupled with conditions (C1) and (C2); moreover, (C4) includes the important and natural choice $\{1/n\}$ for $\{\lambda_n\}$.

Motivated by the above works, in this paper we suggest and analyze a hybrid iterative algorithm. It is shown that the proposed algorithm converges to $x^* \in \bigcap_{n=1}^{\infty} Fix(T_n)$ which solves some variational inequality under some mild assumptions.

2. Preliminaries

In this section, we first recall some notations. Let C be a nonempty closed convex subset of a real Hilbert space H. Let $F : C \to C$ be an operator. F is called k-Lipschitzian if there exists a positive constant k such that

$$\|Fx - Fy\| \leqslant k\|x - y\|,$$

for all $x, y \in C$. F is said to be η-strongly monotone if there exists a positive constant η such that

$$\langle Fx - Fy, x - y\rangle \geqslant \eta\|x - y\|^2,$$

for all $x, y \in C$. Without loss of generality, we can assume that $\eta \in (0, 1)$ and $k \in [1, \infty)$. Under these conditions, it is well-known that the variational inequality problem $VI(F, C)$ has a unique solution $x^* \in C$.

In the sequel, we shall make use of the following well-known Lemmas.

Lemma 2.1. *Let H be a real Hilbert space. There holds the following well-known identity:* $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$, $\forall x, y \in H$.

Lemma 2.2 (Demi-closed Principle). *Assume that $T : H \to H$ is a non-expansive mapping. If T has a fixed point, then I–T is demi-closed. That is, whenever $\{x_n\}$ is a sequence in H weakly converging to some $x \in H$ and the sequence $\{(I - T)x_n\}$ strongly converges to some y, it follows that $(I - T)x = y$. Here I is the identity operator of H.*

3. Main results

In this section, we first introduce our iterative algorithm. Consequently, we will establish strong convergence theorem for this iteration algorithm.

Let $T_1, T_2, \ldots$ be an infinite family of non-expansive mappings of H into itself and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 \leqslant \xi_i \leqslant 1$ for every $i \in N$. For any $n \in N$, define a mapping W_n of H into itself as follows: