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Approximate bound state solutions of Dirac equation with Hulthén potential including Coulomb-like tensor potential

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ABSTRACT

We solve the Dirac equation approximately for the attractive scalar $S(r)$ and repulsive vector $V(r)$ Hulthén potentials including a Coulomb-like tensor potential with arbitrary spin-orbit coupling quantum number κ. In the framework of the spin and pseudospin symmetric concept, we obtain the analytic energy spectrum and the corresponding two-component upper- and lower-spinors of the two Dirac particles by means of the Nikiforov–Uvarov method in closed form. The limit of zero tensor coupling and the non-relativistic solution are obtained. The energy spectrum for various levels is presented for several κ values under the condition of exact spin symmetry in the presence or absence of tensor coupling.

1. Introduction

Within the framework of the Dirac equation the spin symmetry arises if the magnitude of the attractive Lorentz scalar potential $S(r)$ and the time-component repulsive vector potential are nearly equal, $S(r) \sim V(r)$ in nuclei (i.e., when the difference potential $\Delta(r) = V(r) - S(r) = C_s =$ constant). However, the pseudospin symmetry occurs when $S(r) \sim -V(r)$ are nearly equal (i.e., when the sum potential $\sum(r) = V(r) + S(r) = C_{ps} =$ constant) [1–3]. The bound states of nucleons seem to be sensitive to some mixtures of these potentials. The cases $\Delta(r) = 0$ and $\sum(r) = 0$ correspond to $SU(2)$ symmetries of the Dirac Hamiltonian [3]. The spin symmetry is relevant for mesons [4]. The pseudospin symmetry concept has been applied to many systems in nuclear physics and related areas [2–7]. Further, it is used to explain features of deformed nuclei [8], the super-deformation [9] and to establish an effective nuclear shell-model scheme [5,6,10]. The pseudospin symmetry appeared in nuclear physics refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where n, l and j are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively [5,6]. The total angular momentum is given as $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is a pseudo-angular momentum and $\tilde{s} = 1/2$ is a pseudospin angular momentum. In real nuclei, the pseudospin symmetry is only an approximation and the quality of approximation depends on the pseudo-centrifugal potential and pseudospin orbital potential [11]. The Dirac Hamiltonian with vector and scalar potentials quadratic in space coordinates has been studied [12]. It was shown that the the Dirac equation can be solved exactly for the cases $\Delta(r) = 0$ and $\sum(r) = 0$. In addition, the linear tensor potential and mixture of quadratic scalar and vector potentials are studied [13]. It is shown that a linear tensor potential with quadratic $\Delta(r)$ or $\sum(r)$ generates an harmonic oscillator-like second-order differential equation which can be solved analytically. Recently, Akcay [14,15] has shown that the Dirac equation for scalar and vector quadratic potentials including the Coulomb-like tensor potential with the spin and pseudospin symme-

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tries can be solved exactly. These results in Dirac equation with quadratic potential plus a centrifugal-like potential can also be solved analytically. Tensor coupling potentials are added as spin-orbit coupling terms to the Dirac Hamiltonian by making the substitution $\vec{p} \to \vec{p} - im\omega\boldsymbol{\beta}.\hat{r}U(r)$ [16,17]. Tensor couplings and exactly solvable tensor potential have been used to investigate nuclear properties [18–20] and have also some physical applications [21,22].

In the past years, there has been much interest in the solution of the relativistic Dirac and Klein–Gordon equations [1,12,23–26]. For instance, some authors have solved these equations for several physical potentials, such as the Woods–Saxon potential [23–26], the Morse potential [27], the Hulthén potential [28], the Eckart potential [29–31], the Pöschl–Teller potential [32,33] and the Scarf-type potential [34], etc.

Recently, many works have been done to solve the Dirac equation so to obtain the energy equation and the two-component spinor wave functions. Jia et al [35] employed an improved approximation scheme to deal with the new centrifugal spin-orbit term $\kappa(\kappa+1)r^{-2}$ in the second-order differential equation that results from the Dirac equation and to solve it for the generalized Pöschl–Teller potential for arbitrary spin-orbit quantum number κ. Zhang et al [36] solved the Dirac equation with equal Scarf-type scalar and vector potentials by the method of the supersymmetric (SUSY) quantum mechanics, shape invariance approach and by alternative methods. Zou et al [37] solved the Dirac equation with equal Eckart scalar and vector potentials in terms of SUSY quantum mechanical method, shape invariance approach and function analysis method. Wei and Dong [38] obtained approximately the analytical bound state solutions of the Dirac equation with the Manning–Rosen for arbitrary κ. Thylwe [39] presented the approach inspired by amplitude-phase method in analyzing the radial Dirac equation to calculate phase shifts by including the spin- and pseudospin symmetries of relativistic spectra. Alhaidari [40] solved Dirac equation by separation of variables in spherical coordinates for a large class of non-central electromagnetic potentials. Berkdemir and Sever [41] investigated systematically the pseudospin symmetric solution of the Dirac equation for spin 1/2 particles moving within the Kratzer potential connected with an angle-dependent potential. Recently, we have also solved the spin and pseudospin symmetric Dirac equation with arbitrary spin-orbit centrifugal term for generalized Woods–Saxon potential [42] and Rosen–Morse potential [43] by means of the Nikiforov–Uvarov (NU) method.

In this paper, it is worth to investigate the solution of the Dirac equation for scalar and vector Hulthén potential for $\Delta(r)$ or $\sum(r)$ together with Coulomb shape tensor coupling potential which can be solved analytically by using an improved approximation scheme introduced in Refs. [44,45] to deal with the resulting centrifugal and pseudo-centrifugal terms $\kappa(\kappa \pm 1)r^{-2}$. The Coulomb-like tensor potential preserves the form of the Hulthén potential but generates a new spin-orbit centrifugal terms $\Lambda(\Lambda \pm 1)r^{-2}$, where Λ is a new spin-orbit quantum number. This provides a possibility for generating a different form of spin-orbit coupling which might have some physical applications.

The Hulthén potential, widely used for the description of the nucleon-heavy nucleus interactions, takes the following form (see [44] and the references therein):

$$V_H(r) = -\frac{V_0}{e^{r/r_0}-1}, \quad r_0 = \delta^{-1}, \quad V_0 = Ze^2\delta, \tag{1}$$

where V_0 is the potential depth, δ is the screening range parameter and r_0 represents the spatial range. If the potential is used for atoms, then $V_0 = Z\delta$(in the relativistic units $\hbar = c = e = 1$), where Z is identified as the atomic number. The Hulthén potential behaves like the Coulomb potential near the origin (i.e., $r \to 0$ or $r \ll r_0$) $V_C(r) = -Ze^2/r$, but decreases exponentially in the asymptotic region when $r \gg 0$, so its capacity for bound states is smaller than the Coulomb potential. This potential has been applied to a number of areas such as nuclear and particle physics [46–48], atomic physics [49,50], molecular physics [51,52] and chemical physics [53], etc.

In the presence of the spin and pseudospin symmetry, we investigate the bound state energy eigenvalues and corresponding upper and lower spinor wave functions for arbitrary spin-orbit κ quantum number in the framework of the NU method [54–56]. We also show that the spin and pseudospin symmetric Dirac solutions can be reduced to the $S(r) = V(r)$ and $S(r) = -V(r)$ in the cases of exact spin symmetry limitation $\Delta(r) = 0$ and pseudospin symmetry limitation $\sum(r) = 0$, respectively. Furthermore, the solutions obtained for the Dirac equation can be easily reduced to the Schrödinger solutions when a parametric transformation is applied.

In what follows, we first review the NU method and present a parametric generalization in Section 2. In the presence of spin and pseudospin symmetries, we obtain the bound state solutions of the Dirac equation with scalar and vector Hulthén potentials including the Coulomb-like tensor interaction, the limit of zero tensor coupling and the non-relativistic limits by applying a suitable transformation in Section 3. The relevant concluding remarks are given in Section 4.

2. The Nikiforov–Uvarov method

The NU method [54] is briefly outlined here. It is based on solving the second-order differential equation of hypergeometric-type by means of special orthogonal functions:

$$\psi''(r) + \frac{\tilde{\tau}(r)}{\sigma(r)}\psi'(r) + \frac{\tilde{\sigma}(r)}{\sigma^2(r)}\psi(r) = 0, \tag{2}$$

where $\sigma(r)$ and $\tilde{\sigma}(r)$ are polynomials at most of second-degree, $\tilde{\tau}(r)$ is a first-degree polynomial and $\psi(r)$ is function of the hypergeometric-type. In order to find a particular solution for Eq. (2), we choose $\psi(r)$ as follows: