Applied Mathematics and Computation 216 (2010) 2512–2524

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Algebraic $\mathbb{C}^*$-actions and the inverse kinematics of a general 6R manipulator

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ARTICLE INFO

Keywords:
Homotopy methods
Continuation
Polynomial systems
Kinematics

ABSTRACT

Let X be a smooth quadric of dimension $2m$ in $\mathbb{P}^{2m+1}_{\mathbb{C}}$ and let $Y, Z \subset X$ be subvarieties both of dimension m which intersect transversely. In this paper we give an algorithm for computing the intersection points of $Y \cap Z$ based on a homotopy method. The homotopy is constructed using a $\mathbb{C}^*$-action on X whose fixed points are isolated, which induces Bialynicki-Birula decompositions of X into locally closed invariant subsets. As an application we present a new solution to the inverse kinematics problem of a general six-revolute serial-link manipulator.

1. Introduction

This paper introduces a homotopy construction for computing numerical approximations to the intersection of two m-dimensional algebraic subsets of a smooth $2m$ dimensional quadric. This new method joins the larger family of homotopy techniques, also known as continuation methods, which have proven to be effective for numerically solving systems of polynomial equations [10,16]. These methods provide a means of constructing a homotopy function and a finite set of start points such that the paths emanating from the start points end in a finite set of endpoints that contain all isolated solutions of the equations. For efficiency, it is desirable that the number of homotopy paths is as small as possible, preferably equal to the actual number of isolated solutions.

Over the years, the pursuit of reduction in the number of homotopy paths has led to a series of homotopy constructions, each successively recognizing more of the structure of the given polynomials. Notable milestones are total degree homotopies [4], multihomogeneous formulations [11], linear set structures [19], and polyhedral homotopies [6,20]. The latter completely accounts for the sparse structure of the monomials in the system, but requires the computation of the mixed volume of the associated Newton polytopes, a combinatorial problem. Nevertheless, the approach is general and can be completely automated. Even the polyhedral homotopies may require more than the minimal number of paths, as in practice, the coefficients of a polynomial system may have interrelations that reduce the number of isolated roots compared to a system with the same monomials but general coefficients. Parameter homotopies [12,16] capture the coefficient relations, but require an initial solution of a generic problem in the parameterized family, which is usually obtained by one of the aforementioned general techniques. More recently, techniques have been introduced for solving systems by introducing the equations one at a time [15,5]. This often has the effect of revealing structure at early stages of the computation, when it is inexpensive

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doi:10.1016/j.amc.2009.12.014

to work with fewer variables and equations, thereby reducing the number of paths and the cost in the final, most expensive, stage involving all the equations. These methods do not incur the cost of the mixed volume computation and they may take advantage of coefficient relations. For some large, sparse systems, the regeneration equation-by-equation method [5] outperforms the polyhedral approach even though it uses more solution paths. Several computer codes [2,9,17,18,21,22] are available that implement one or more of the homotopies just mentioned.

The method presented in this paper resembles the equation-by-equation approaches in that less expensive preliminary computations can reveal structure that reduces the path count, and hence the computational expense, of the final homotopy. The technique is based on the cell decomposition of a quadric induced by a multiplicative $\mathbb{C}^*$-action. This $\mathbb{C}^*$-homotopy applies when one seeks the isolated points in the intersection of two m-dimensional algebraic subsets of a $2m$-dimensional smooth quadric in $\mathbb{P}^{2m+1}_{\mathbb{C}}$. While this is not as general as the techniques previously mentioned, the situation arises often in applications, where quadrics are frequently encountered. The method has the desirable property that it subdivides the target problem in $2m+1$ dimensions into four subproblems, each in only m-dimensions. The solutions to these subproblems are combined to form the start points for a final homotopy that solves the target problem. It may happen that one or more of the subproblems has fewer solutions than its total degree would suggest, in which case the final homotopy has fewer than the total degree number of paths.

This work was inspired by a geometrical problem from robotics: the inverse kinematics of a general six-revolute (6R) serial-link robot. The objective in inverse kinematics is to find all sets of joint angles that place the end-effector of a robot in a desired location. For general 6R robots, that is, for robots not having certain simplifying geometries such as intersecting wrist axes, it has been known since 1986 [13] that the problem has 16 solutions (over the complex number field). The early proofs and the related algorithms for calculating the joint angles depend on rather intricate algebraic manipulations of the defining polynomial equations. However, in 2005, Selig [14, Section 11.5] gave a simple, although abstract, proof based on intersection theory and a cell decomposition of the Study quadric, an elegant representation of $SE(3)$, the space of rigid-body displacements.

In the work reported here, we turn Selig's abstract proof into a concrete homotopy method for numerically solving the inverse kinematics problem using just 16 paths in the final homotopy to find the 16 solutions. As the Study quadric is fundamental to robotics, we expect that an algorithm for 6R inverse kinematics that makes strong use of the properties of the Study quadric might lead to better insight on solving other problems in robot kinematics. In fact, as outlined above, our pursuit of the 6R problem has lead to a solution algorithm that applies much more generally than to robot kinematics.

This paper is organized as follows. We begin in Section 2 by describing the $\mathbb{C}^*$-action on a quadric, that is central to our homotopy construction, and by presenting the cell decomposition that it induces. In doing so, we introduce the notation used throughout the paper. After a brief review, in Section 3, of some basic ideas in continuation, Section 4 presents the homotopy construction and the method of determining start points for the homotopy. The original statement of the algorithm in Section 4.1 is made for intersecting algebraic sets determined implicitly by polynomial equations, while in Section 4.2 the method is extended to cover the case where the sets are defined parametrically. In Section 5 we show the application of the method to the 6R inverse kinematics problem.

2. C^*-Actions and cell decomposition

Let X be a smooth quadric hypersurface of even dimension $2m$ in the projective space $\mathbb{P}^{2m+1}$ over $\mathbb{C}$. Let $[q_0,\ldots,q_m,p_0,\ldots,p_m]$ be homogeneous coordinates[1] on $\mathbb{P}^{2m+1}$. We may assume that X is defined by the equation

$$Q(q,p) = q_0p_0 + q_1p_1 + \cdots + q_mp_m = 0. \tag{1}$$

This is because any smooth quadric is given by a polynomial of the form $x^TAx = 0$, where A is a nonsingular, symmetric matrix. Hence, A can be written as $A = A^{1/2}A^{1/2}$, where $A^{1/2}$ is a symmetric matrix with inverse $A^{-1/2}$. Let N be the matrix

$$N = \begin{bmatrix} I & I \\ -Ii & Ii \end{bmatrix}, \tag{2}$$

and let $q = [q_0 \quad \cdots \quad q_m]$ and $p = [p_0 \quad \cdots \quad p_m]$ be row vectors. Then one may make the nonsingular change of coordinates $x = A^{-1/2}N[q,p]^T$, where upon

$$x^TAx = [q,p]N^TN[q,p]^T = 4Q(q,p).$$

So $x^TAx = 0$ implies that $Q(q,p) = 0$.

We fix an action of the multiplicative group $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$ on X defined as follows: for $t \in \mathbb{C}^*$

$$t[q_0,\ldots,q_m,p_0,\ldots,p_m] = [q_0, tq_1,\ldots,t^mq_m, t^{2m}p_0, t^{2m-1}p_1,\ldots,t^mp_m]. \tag{3}$$

[1] We use square brackets $[\ldots]$ to denote homogeneous coordinates. Each point in $\mathbb{P}^n$ corresponds to a line through the origin in $\mathbb{C}^{n+1}$. For $y,z \in \mathbb{P}^n$, the equality $y = z$ means that the corresponding homogeneous coordinates in $\mathbb{C}^{n+1}$ are equal up to a nonzero scalar.