Applied Mathematics and Computation 215 (2010) 3214–3221

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The extended $\left(\frac{G'}{G}\right)$-expansion method and its applications to the Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations

Shimin Guo, Yubin Zhou *

*School of Mathematics and Statistics, Lan Zhou University, Lan Zhou 730000, China*

ARTICLE INFO

*Keywords:*
Extended $\left(\frac{G'}{G}\right)$-expansion method
Solitary wave solution
Nonlinear evolution equations
Whitham–Broer–Kaup–Like equations
Coupled Hirota–Satsuma KdV equations

ABSTRACT

In this paper, based on a new general ansätze, the extended $\left(\frac{G'}{G}\right)$-expansion method is proposed to seek exact solutions of nonlinear evolution equations. Being concise and straightforward, this method is applied to construct travelling wave solutions of Whitham–Broer–Kaup–Like equations and coupled Hirota–Satsuma KdV equations. By using this method, new exact solutions involving parameters, expressed by three types of functions which are the hyperbolic functions, the trigonometric functions and the rational functions, are obtained. When the parameters are taken as special values, some solitary wave solutions are derived from the hyperbolic function solutions.



## 1. Introduction

In the present paper, we will seek exact solutions of the following two nonlinear evolution equations (NLEEs): one is the Whitham–Broer–Kaup–Like (WBKL) equations [1] in the form

$$\begin{cases} u_t + uu_x + \gamma H_x + \beta u_{xx} = 0, \\ H_t + (Hu)_x + \alpha u_{xxx} - \beta H_{xx} = 0, \end{cases} \tag{1.1}$$

where $\alpha, \beta, \gamma$ are real constants, and another is the coupled Hirota–Satsuma KdV (CHSK) equations [2] in the form

$$\begin{cases} u_t = \frac{1}{4}u_{xxx} + 3uu_x - 6vv_x, \\ v_t = -\frac{1}{2}v_{xxx} - 3uv_x, \end{cases} \tag{1.2}$$

which is first proposed by Hirota–Satsuma in 1981 [3]. It is necessary to point out that when the parameters are taken as different values, the following celebrated NLEEs can be derived from Eq. (1.1).

(i) When $\gamma = 1$, we have the Whitham–Broer–Kaup equations [4]

$$\begin{cases} u_t + uu_x + H_x + \beta u_{xx} = 0, \\ H_t + (Hu)_x + \alpha u_{xxx} - \beta H_{xx} = 0. \end{cases} \tag{1.3}$$

(ii) When $\alpha = 0, \gamma = 1$, we get the approximate equations for long wave equations

* Corresponding author.
*E-mail address:* mybzhou@yahoo.com.cn (Y. Zhou).

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.10.008

$$\begin{cases} u_t + uu_x + H_x + \beta u_{xx} = 0, \\ H_t + (Hu)_x - \beta H_{xx} = 0. \end{cases} \tag{1.4}$$

(iii) When $\alpha = \gamma = 1, \beta = 0$, we obtain the variant Boussinesq equations [5,6]

$$\begin{cases} u_t + uu_x + H_x = 0, \\ H_t + (Hu)_x + u_{xxx} = 0. \end{cases} \tag{1.5}$$

(iv) When $\alpha = \frac{1}{3}, \beta = 0, \gamma = 1$, we get the dispersive long wave equations [7,8]

$$\begin{cases} u_t + uu_x + H_x = 0, \\ H_t + (Hu)_x + \frac{1}{3} u_{xxx} = 0. \end{cases} \tag{1.6}$$

It is clear to see that both Eqs. (1.1) and (1.2) are very important in the field of mathematical physics. Therefore, it is a significant task to search for explicit solutions of the two equations. Up to now, many powerful methods have been established and developed to obtain analytic solutions of NLEEs, such as the inverse scattering method [9], the truncated Painlevé expansion [10], the Bäcklund transformation [11], the Darboux transformation [12], the Lie symmetries method [13], the homogenous balance method [14–16], and so on.

In recent years, direct methods [17–28] to construct exact solutions of NLEEs have become more and more attractive partly due to the availability of symbolic computation packages like Maple and Mathematica, which enable us to perform the tedious and complex computation on computer. One of the most effective direct methods to construct travelling wave solutions of NLEEs is the $(\frac{G'}{G})$-expansion method, which was first proposed by Wang et al. in Ref. [29]. The $(\frac{G'}{G})$-expansion method is based on the assumptions that the travelling wave solutions of NLEEs can be expressed by a polynomial in $(\frac{G'}{G})$, where $G = G(\xi)$ satisfies the following second order linear ordinary differential equation (LODE)

$$G'' + \lambda G' + \mu G = 0, \tag{1.7}$$

where $G'' = \frac{d^2 G(\xi)}{d\xi^2}, G' = \frac{dG(\xi)}{d\xi}, \xi = x - Vt, \lambda, \mu$ and $V$ are constants. The degree of the polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms in the given NLEEs, and the coefficients of the polynomial can be obtained by solving a set of algebraic equations resulted from the process of using the $(\frac{G'}{G})$-expansion method. By means of this method, Wang et al. successfully obtained more travelling wave solutions of four NLEEs in Ref. [29]. As we know, the choice of an appropriate ansätz is of great importance when using the direct methods. In this paper, based on a new general ansätze, we propose the extended $(\frac{G'}{G})$-expansion method, which can be used to obtain explicit solutions of NLEEs.

The rest of this paper is organized as follows: In the following Section 2, a general theory of the extended $(\frac{G'}{G})$-expansion method for finding out exact solutions of NLEEs is described. In Section 3, we illustrate the applications of this method to the Whitham–Broer–Kaup–Like Eq. (1.1) and coupled Hirota–Satsuma KdV Eq. (1.2). In Section 4, some conclusions are given.

## 2. Description of the extended $(\frac{G'}{G})$-expansion method

Suppose that a nonlinear evolution equation, say in two independent variables $x$ and $t$, is given by

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt} \ldots,) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, $P$ is a polynomial in $u = u(x,t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of the extended $(\frac{G'}{G})$-expansion method.

**Step 1** : Use the travelling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = x - Vt, \tag{2.2}$$

where $V$ is a constant to be determined latter. Then, the NLEE (2.1) is reduced to the following nonlinear ordinary differential equation (NLODE) for $u = u(\xi)$:

$$P(u, -Vu', u', V^2 u'', u'', -Vu'' \ldots) = 0. \tag{2.3}$$

**Step 2** : We suppose that the NLODE (2.3) has the following solution:

$$u(\xi) = a_0 + \sum_{i=1}^{n} \left\{ a_i \left(\frac{G'}{G}\right)^i + b_i \left(\frac{G'}{G}\right)^{i-1} \sqrt{\sigma \left(1 + \frac{1}{\mu}\left(\frac{G'}{G}\right)^2\right)} \right\}, \tag{2.4}$$