Applied Mathematics and Computation 215 (2010) 3482–3487

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Invariants and approximate solutions for certain non-linear oscillators by means of the field method

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ARTICLE INFO

Keywords:
Field method
Convolution integral
Adiabatic invariants
Approximate solution

ABSTRACT

Certain strongly non-linear conservative oscillators are approached with the field method, which is combined with the convolution integral method. A complete set of their adiabatic invariants are derived, on the basis of which approximate solutions for motion can be obtained.

1. Introduction

It is generally accepted that invariants (conserved quantities) of mechanical systems provide useful insights into the qualitative properties of their dynamics. As far as their classification is concerned, invariants can be classified as 'exact' (conservation laws, constants of motion) and 'adiabatic' (approximate conservation laws, approximate invariants). According to the definition given in [1], some function represents an adiabatic invariant to $O(\varepsilon^N)$ if its total derivative, by virtue of the equation of motion, is of $N+1$ order with respect to a small parameter ε. If that derivative is equal to zero, the function represents the exact invariant. The invariants can also be broadly classified according to the degree of the polynomial form in momenta or velocities as linear, quadratic, cubic, etc. [2].

A great number of various approaches has been aimed at obtaining adiabatic invariants of mechanical oscillators. It was Burgers [3] who pioneered research in adiabatic invariants. Many other studies in after years have contributed to this field [4–10], attesting to the theoretical and practical importance of the knowledge of these quantities. In spite of considerable attention which has been paid to finding them, there still exists interest in deriving adiabatic invariants, especially for the systems modelled by non-linear differential equations.

In this study, our aim is to obtain adiabatic invariants of non-linear autonomous oscillators:

$$\ddot{x} + G(x) = 0, \tag{1}$$

$$x(0) = a, \quad \dot{x}(0) = 0, \tag{2}$$

where $G(x)$ is an odd function of a coordinate x, which does not necessarily have a linear term and overdots denote differentiation with respect to time t. The problem is approached by the field method technique [11,12], which has been approved as beneficial for studying different problems of disparate areas of mechanics [11,13–17]. In this approach it results in a complete set of linear time-dependent invariants of convolution type. Their combination leads to a quadratic adiabatic invariant, whose correspondence with an exact invariant is discussed. Namely, it is sometimes possible to construct approximate invariants even for the systems admitting exact invariants, which helps in finding the solution of the problem, although in an approximate manner [2]. Thus, having found the complete set of the adiabatic invariants of the system (1), its approximate solution of motion is derived. In comparison to many techniques for the construction of the analytical approximations

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.10.025

to the non-linear oscillators (1) [20–23], the proposed procedure gives the approximate solution for motion as well as new results regarding the qualitative properties of the system being considered in the form of its adiabatic invariants.

2. Field method approach

In order to apply the field method algorithm developed for obtaining conservation laws of the linear one-degree of freedom oscillators [12], the system (1) can be written down as:

$$\dot{x} = p, \quad \dot{p} = -\omega^2 x + F, \tag{3}$$

where

$$F \equiv F(t) = \omega^2 x(t) - G(x(t)), \tag{4}$$

and ω is the frequency to be found. Then, the basic assumption of the field method will be introduced, which is that the coordinate x can be represented as a field depending on time t and the momentum (i.e. the velocity) p

$$x = U(t,p). \tag{5}$$

Partial differentiation of the expression (5) in combination with (3) yields

$$\frac{\partial U}{\partial t} + \frac{\partial U}{\partial p}[-\omega^2 U + F(t)] - p = 0. \tag{6}$$

The solution of this partial differential equation can be assumed in the form [12,17–19]:

$$U = Ap + f(t), \tag{7}$$

with A being a constant and $f(t)$ being an unknown function of time. Substituting it into (6) and equating the terms involving p and the free terms with zero, one has:

$$A_{1/2} = \pm\frac{i}{\omega}, \tag{8}$$

$$f(t) = Ce^{A\omega^2 t} - Ae^{A\omega^2 t}\int_0^t F(\tau)e^{-A\omega^2\tau}d\tau, \tag{9}$$

where i is an imaginary unit and C is a constant.

For two values of the constant A (8), algebraic transformations of the assumed form (7) lead to the expressions in which the convolution integrals [24] appear:

$$x - \frac{ip}{\omega} + \frac{i}{\omega}\int_0^t [\omega^2 x(\tau) - G(x(\tau))]e^{-i\omega(\tau-t)}d\tau = C_1 e^{i\omega t}, \tag{10}$$

$$x + \frac{ip}{\omega} - \frac{i}{\omega}\int_0^t [\omega^2 x(\tau) - G(x(\tau))]e^{i\omega(\tau-t)}d\tau = C_2 e^{-i\omega t}. \tag{11}$$

To solve the integrals in (10) and (11), the solution for the coordinate x inside the square brackets is assumed as $x(\tau) \approx \frac{C_1 e^{i\omega\tau} + C_2 e^{-i\omega\tau}}{2}$. In accordance with the initial conditions (2), this form gives the constants C_1 and C_2:

$$C_1 = C_2 = a. \tag{12}$$

Now, the expressions (10) and (11) can be presented as:

$$\left[x - \frac{ip}{\omega}\right]e^{-i\omega t} + \frac{i}{\omega}\int_0^t \left[\omega^2\frac{a + ae^{-i2\omega\tau}}{2} - e^{-i\omega\tau}G\left(\frac{ae^{i\omega\tau} + ae^{-i\omega\tau}}{2}\right)\right]d\tau = a, \tag{13}$$

$$\left[x + \frac{ip}{\omega}\right]e^{i\omega t} - \frac{i}{\omega}\int_0^t \left[\omega^2\frac{ae^{i2\omega\tau} + a}{2} - e^{i\omega\tau}G\left(\frac{ae^{i\omega\tau} + ae^{-i\omega\tau}}{2}\right)\right]d\tau = a, \tag{14}$$

the frequency ω will be calculated from the elimination of secular terms among the terms generated by the integrals. Eliminating the secular terms and integrating the remaining terms, some function of time will be obtained. Together with the terms in front of the integrals, they will form two independent linear adiabatic invariants. They provide additional information about the behavior of the system being considered, giving us the combinations of the parameters of the system which remain almost constant during time. Besides, they enable us to find a quadratic approximate invariant as their product. It can be presented in the form:

$$x^2 + \frac{p^2}{\omega^2} + xD_1(t) + pD_2(t) + D_3(t) = a^2, \tag{15}$$

where $D_1(t), D_2(t)$ and $D_3(t)$ stand for some functions of time. Depending explicitly on time, the quadratic form (15) differs from the corresponding exact invariant (total energy conservation law) of the system (1):