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On nonlinear two-point boundary value problems for impulsive differential-algebraic problems

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ABSTRACT

In this paper, we investigate boundary value problems for first order impulsive differential-algebraic problems with causal operators. Note that a corresponding boundary condition is given by a nonlinear function. Using a monotone iterative method we formulate general sufficient conditions under which such problems have solutions (extremal or a unique). An example shows that corresponding assumptions are satisfied. The results are new.

1. Introduction

Let $J = [0, T],\ E = C(J, \mathbb{R})$ and $Q \in C(E, E)$. We shall say that Q is a causal operator, or nonanticipative, if the following property holds: for each couple of elements of E such that $u(s) = v(s)$ for $0 \leqslant s \leqslant t$, there results $(Qu)(s) = (Qv)(s)$ for $0 \leqslant s \leqslant t$ with $t < T$ arbitrary, for details see [3].

Let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$. Put $J' = J \setminus \{t_1, t_2, \ldots, t_m\}$. In this paper, we investigate nonlinear two-point boundary value problems for impulsive functional differential-algebraic equations with causal operators Q_1, Q_2 of the form

$$\begin{cases} x'(t) = (Q_1(x, y))(t), & t \in J', \\ \Delta x(t_k) = I_k(x(t_k)), & k = 1, 2, \ldots, m, \\ 0 = g(x(0), x(T)), \\ y(t) = (Q_2(x, y))(t), \end{cases} \tag{1}$$

where as usual $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$; $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k, respectively.

Differential equations with causal operators are discussed in book [5], see also the references therein. To find approximate solutions of nonlinear differential problems we can use the monotone iterative method, for details see for example [22]. We have many applications of this technique to boundary value problems of nonlinear (impulsive) differential equations, see for example [4,6,10,12–16,21,23–26], see also [1,27–29]. Recently, this technique was also applied to problems with causal operators, see for example [7–9,17–20].

Differential-algebraic equations appeared in mathematical models of various problems of physics, engineering, chemistry and other branches of sciences, see [3]. Quite general classes of integro-algebraic systems and differential-algebraic systems are investigated in paper [2]; see also [11]. In this paper, we extend the application of the monotone iterative technique to find solutions (extremal) of nonlinear two-point boundary value problems for differential-algebraic equations with causal operators.

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2. Linear impulsive differential inequalities

Put $J_0 = [0, t_1],\ J_k = (t_k, t_{k+1}],\ k = 1, 2, \ldots, m$. Let us introduce the spaces:

$$PC(J) = PC(J, \mathbb{R}) = \left\{ \begin{array}{l} x : J \to \mathbb{R},\ x|_{J_k} \in C(J_k, \mathbb{R}),\ k = 0, 1, \ldots, m \\ \text{and there exist } x(t_k^+) \text{ for } k = 1, 2, \ldots, m \end{array} \right\}$$

and

$$PC^1(J) = PC^1(J, \mathbb{R}) = \left\{ \begin{array}{l} x \in PC(J),\ x|_{J_k} \in C^1(J_k, \mathbb{R}),\ k = 0, 1, \ldots, m \\ \text{and there exist } x'(t_k^+) \text{ for } k = 1, 2, \ldots, m \end{array} \right\}.$$

We shall first concentrate our attention to differential inequalities with positive linear operators. We shall say that a linear operator $\mathscr{L} \in C(E, E)$ is a positive linear operator if $(\mathscr{L}m)(t) \geqslant 0$ provided that $m(t) \geqslant 0$ on J. We need the following

Lemma 1 (see [18]). *Let $\mathscr{L} \in C(E, E)$ be a positive linear operator. Let $K \in C(J, \mathbb{R}), L_k \in [0, 1),\ k = 1, 2, \ldots, m$. Let $p \in PC^1(J, \mathbb{R})$ and*

$$\begin{cases} p'(t) \leqslant -K(t)p(t) - (\mathscr{L}p)(t), & t \in J', \\ \Delta p(t_k) \leqslant -L_k p(t_k), & k = 1, 2, \ldots, m, \\ p(0) \leqslant r p(T), & 0 \leqslant r \leqslant e^{\int_0^T K(s)ds}. \end{cases}$$

In addition, we assume that

$$\int_0^T e^{\int_0^s K(\tau)d\tau} (\mathscr{L}\bar{p})(s)ds + \sum_{i=1}^m L_i \leqslant 1 \quad \textit{with } \bar{p}(t) = e^{-\int_0^t K(\tau)d\tau}. \tag{2}$$

Then $p(t) \leqslant 0,\ t \in J$.

In Lemma 1, it is assumed that $K \in C(J, \mathbb{R})$ and r is bounded by $0 \leqslant r \leqslant \rho \equiv e^{\int_0^T K(s)ds}$. Note that ρ depends on K, *and* ρ may be bigger or less than 1. Condition (2) is important in Lemma 1. If we assume that $K \in C(J, \mathbb{R}_+)$ and $0 \leqslant r \leqslant 1$, then, in the place of condition (2), we can also obtain another condition. This case is discussed in the next lemma.

Lemma 2. (see [18]) *Let $\mathscr{L} \in C(E, E)$ be a positive linear operator. Let $K \in C(J, \mathbb{R}_+)$, $L_k \in [0, 1),\ k = 1, 2, \ldots, m$. Let $p \in PC^1(J, \mathbb{R})$ and*

$$\begin{cases} p'(t) \leqslant -K(t)p(t) - (\mathscr{L}p)(t), & t \in J', \\ \Delta p(t_k) \leqslant -L_k p(t_k), & k = 1, 2, \ldots, m, \\ p(0) \leqslant r p(T), & 0 \leqslant r \leqslant 1. \end{cases}$$

In addition, we assume that

$$\int_0^T [K(s) + (\mathscr{L}1)(s)]ds + \sum_{i=1}^m L_i \leqslant 1. \tag{3}$$

Then $p(t) \leqslant 0,\ t \in J$.

3. Linear impulsive differential equations

Now we consider the following impulsive problem:

$$\begin{cases} v'(t) = -K(t)v(t) - (\mathscr{L}v)(t) + \eta(t), & t \in J', \\ v(t_k^+) = (1 - L_k)v(t_k) + \gamma_k, & k = 1, 2, \ldots, m, \\ v(0) = r v(T) + \beta, & \beta \in \mathbb{R},\ 0 \leqslant r. \end{cases} \tag{4}$$

The next theorem concerns conditions under which problem (4) has a unique solution.

Theorem 1 (see [18]). *Let $K \in C(J, \mathbb{R}),\ \eta \in PC(J),\ L_k \in [0, 1),\ \gamma_k \in \mathbb{R},\ k = 1, 2, \ldots, m$. Let $\mathscr{L} \in C(E, E)$ be a positive linear operator and let $r_1 = re^{-\int_0^T K(s)ds} \neq 1$. In addition, we assume that $\rho_1 < 1$ with*

$$\rho_1 = \sup_t \frac{e^{-\int_0^t K(s)ds}}{|1 - r_1|} \left[r_1 \int_t^T e^{\int_0^s K(\tau)d\tau} (\mathscr{L}1)(s)ds + \int_0^t e^{\int_0^s K(\tau)d\tau} (\mathscr{L}1)(s)ds + r_1 \sum_{i=k+1}^m{}' L_i e^{\int_0^{t_i} K(s)ds} + \sum_{i=1}^k{}' L_i e^{\int_0^{t_i} K(s)ds} \right]. \tag{5}$$

Then problem (4) has a unique solution $v \in PC^1(J)$.