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On positive definite solutions of nonlinear matrix equation $X^s - A^*X^{-t}A = Q$ ☆

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ABSTRACT

In this paper, the nonlinear matrix equation $X^s - A^*X^{-t}A = Q$ is investigated. Based on the fixed-point theory, the existence and the uniqueness of the positive definite solution are studied. An effective iterative method to obtain the unique positive definite solution is established given $\|A\| \cdot \|Q^{-1}\|^{\frac{s+t}{s}} < \frac{s}{t}$. In addition, some computable estimates of the unique positive definite solution are derived. Finally, numerical examples are given to illustrate the effectiveness of the algorithm and the perturbation estimates.

1. Introduction

Consider the nonlinear matrix equation

$$X^s - A^*X^{-t}A = Q, \tag{1.1}$$

where A is a nonsingular n square complex matrix and Q is a positive definite matrix. Both of s and t are positive integers. A^* stands for the conjugate transpose of the matrix A.

This type of matrix equation often arises from many areas such as ladder networks [1,2], dynamic programming [3], control theory [4,5] , stochastic filtering and statistics, etc., [6–8]. Several authors [9–17] have considered such a nonlinear matrix equation problem for special s and t. In [9] the case $s = t = 1$, $Q = I$ is considered and different iterative methods for computing the positive definite solutions are proposed. In addition, for $X - A^*X^{-1}A = Q$, some perturbation estimates of the positive definite solutions are derived in [10,11]. The case $s = 1, t = 2$ and $Q = I$ has been studied in references [12–14]. In [15–17] the authors considered the matrix equation $X - A^*X^{-n}A = Q$ and gave some perturbation bounds for the positive definite solutions. [18,19] studied the positive definite solutions of the matrix equations $X \pm A^*X^{-q}A = Q$ where $q \in (0, 1]$. In [20–22], the authors treat the matrix equations $X + A^*\mathscr{F}(X)A = Q$ where $\mathscr{F}$ maps positive definite matrices either into positive definite matrices or into negative definite matrices, and $\mathscr{F}$ satisfies some monotonicity property. The first attempt to consider the general matrix equation $X^s \pm A^*X^{-t}A = I$ was made in [23] by X.G.Liu and H.Gao for $A \in \mathscr{R}^{n,n}$.

In this paper, we discuss the more general case, namely, $X^s - A^*X^{-t}A = Q$. We study the existence and uniqueness of the positive definite solutions of matrix Eq. (1.1). Based on Banach's fixed-point principle, we give an effective iterative method to obtain the unique positive definite solution when $\|A\|^2 \cdot \|Q^{-1}\|^{\frac{s+t}{s}} < s/t$. Furthermore, for $s < t$, we derive some elegant estimates of the positive definite solutions. Finally, numerical examples are given to illustrate the effectiveness of the algorithm and the perturbation estimates given in this paper.

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We start with some notations which we use throughout this paper. We use $\|\cdot\|$ to denote the spectral norm, and $\|\cdot\|_F$ is the Frobenius norm; $\lambda_{\mathrm{m}}(H)$ and $\lambda_1(H)$ stand for the minimal and maximal eigenvalues of the Hermitian matrix H, respectively. Similarly, $\sigma_{\mathrm{m}}(H)$ and $\sigma_1(H)$ denote the minimal and the maximal singular values of H. For $n\times n$ complex matrix $A=(a_1,a_2,\ldots,a_n)=(a_{ij})$ and a matrix B, $A\otimes B=(a_{ij}B)$ is a Kronecker product; $\mathrm{vec}(A)$ is a vector defined by $\mathrm{vec}(A)=(a_1^T,a_2^T,\ldots,a_n^T)^T$. The notation $X\geqslant Y(X>Y)$ means that X and Y are Hermitian matrices and X- Y is a positive semi-definite(definite) matrix.

2. The positive definite solutions

In this section, we discuss the existence, uniqueness and estimates of the positive definite solutions of matrix Eq. (1.1) and derive an iterative method to obtain the unique positive definite solution given $\|A\|^2\cdot\|Q^{-1}\|^{\frac{s+t}{s}}<s/t$.

Theorem 2.1. *Given an $n\times n$ complex matrix A. Eq. (1.1) always has a positive definite solution X.*

Proof. Let $Y=X^s$. Then Eq. (1.1) is equivalent to the following matrix equation

$$Y-A^*Y^{-t/s}A=Q, \tag{2.1}$$

which is of the form $Y+A^*\mathbf{F}(Y)A=Q$ with $\mathbf{F}(Y)=-Y^{-t/s}$. Let $B=Q+\|Q^{-1}\|^{\frac{t}{s}}A^*A$, then from lemma 2.2 in [20], we know that the matrix Eq. (2.1) has positive definite solution, which completes the proof. □

Let X be any positive definite solution of matrix Eq. (1.1), i.e. $X^s-A^*X^{-t}A=Q$, then it is not difficult to verify that

$$\lambda_m^{\frac{1}{s}}(Q)I\leqslant X\leqslant\left[\lambda_1(Q)+\frac{1}{\lambda_m^{t/s}(Q)}\|A\|^2\right]^{\frac{1}{s}}I=\left[\lambda_1(Q)+\|Q^{-1}\|^{\frac{t}{s}}\|A\|^2\right]^{\frac{1}{s}}I. \tag{2.2}$$

In fact, we have the following estimate which strengthens (2.2).

Theorem 2.2. *The positive definite solution X of $X^s-A^*X^{-t}A=Q$ satisfies*

$$\alpha I\leqslant X\leqslant\beta I,$$

where α and β are positive solutions of the equations

$$\begin{cases}\alpha^s=\lambda_m(Q)+\frac{1}{\beta^t}\sigma_m^2(A)\\ \beta^s=\lambda_1(Q)+\frac{1}{\alpha^t}\sigma_1^2(A).\end{cases} \tag{2.3}$$

Proof. Define the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ as follows:

$$\begin{cases}\alpha_0=\lambda_m^{\frac{1}{s}}(Q),\quad \beta_0=[\lambda_1(Q)+\frac{1}{\alpha_0^t}\sigma_1^2(A)]^{\frac{1}{s}}\\ \alpha_k=[\lambda_m(Q)+\frac{1}{\beta_{k-1}^t}\sigma_m^2(A)]^{\frac{1}{s}}\\ \beta_k=[\lambda_1(Q)+\frac{1}{\alpha_k^t}\sigma_1^2(A)]^{\frac{1}{s}}.\end{cases} \tag{2.4}$$

Similar to the proof of Theorem 3.3 in [18], one can show that

(i). The sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are monotonically increasing and monotonically decreasing, respectively.
(ii). For any positive definite solution X, $X\in[\alpha_kI,\beta_kI]$ for each $k=0,1,2,\ldots\ldots$:

Consequently, the sequences $\{\alpha_k\}$ and $\{\beta_k\}$ are convergent. Denote $\alpha=\lim_{k\to\infty}\alpha_k$, $\beta=\lim_{k\to\infty}\beta_k$, then

$$\alpha I\leqslant X\leqslant\beta I.$$

Taking limits in (2.4), we know that $\alpha,\ \beta$ satisfy (2.3). □

Remark 2.1. Let $Q=I$ in Theorems 2.1 and 2.2, then we get Theorem 2.3 and Theorem 2.5 in [23], respectively; Also we can see that Theorems 2.2 and 2.3 in [15] are special cases of our results when $s=1$, $t=n$ and $Q=I$.

Let A be real and $Q=I$. In [23], the authors gave the following recurrence:

$$\begin{cases}X_{k+1}=[I+A^TX_k^{-t}A]^{\frac{1}{s}}\\ X_0=I.\end{cases}$$

and proved that if $\|A\|^2<s/t$, then the iteration above converges to the unique symmetric positive definite solution of the matrix equation $X^s-A^TX^{-t}A=I$. In general, for matrix Eq. (1.1), we have the following theorem: