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# A system of mixed equilibrium problems, fixed point problems of strictly pseudo-contractive mappings and nonexpansive semi-groups

Shih-sen Chang [a,*], Chi Kin Chan [b], H.W. Joseph Lee [b], Li Yang [c]

[a] Department of Mathematics, Yibin University, Yibin, Sichuan 644007, PR China
[b] Department of Applied Mathematics, The Hong Kong university, Hong Kong
[c] Department of Mathematics, South West University of Science and Technology, Mianyang, Sichuan 621002, PR China

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ABSTRACT

The purpose of this paper is to introduce an iterative algorithm for finding a common element of the set of solutions for a system of mixed equilibrium problems, the set of common fixed points for an infinite family of strictly pseudo-contractive mappings and the set of common fixed points for nonexpansive semi-groups in Hilbert space. Under suitable conditions some strong convergence theorem are proved. The results presented in the paper extend and improve some recent results.



## 1. Introduction

Throughout this paper, we always assume that $H$ is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, respectively, $C$ is a nonempty closed convex subset of $H$ and $P_C$ is the metric projection of $H$ onto $C$. In the sequel, we denote by "$\rightarrow$" and "$\rightharpoonup$" the strong convergence and weak convergence, respectively.

Recall that a mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \forall x, y \in C.$$

We denote by $F(T)$ the set of fixed points of the mapping $T$.

Let $\phi : C \to \mathscr{R}$ be a real-valued function and $\{\Theta_i : C \times C \to \mathscr{R},\ i = 1, 2, \ldots, N\}$ be a finite family of equilibrium bifunctions, i.e., $\Theta_i(u, u) = 0$ for each $u \in C$. The "so-called" *system of mixed equilibrium problems* (SMEP) *for functions* $(\Theta_1, \Theta_2, \ldots, \Theta_N, \phi)$ is to find a common element $x^* \in C$ such that

$$\begin{cases} \Theta_1(x^*, y) + \phi(y) - \phi(x^*) \geqslant 0, & \forall y \in C, \\ \Theta_2(x^*, y) + \phi(y) - \phi(x^*) \geqslant 0, & \forall y \in C, \\ \vdots & \\ \Theta_N(x^*, y) + \phi(y) - \phi(x^*) \geqslant 0, & \forall y \in C. \end{cases} \tag{1.1}$$

We denote the set of solutions of (1.1) by $\Omega := \bigcap_{i=1}^N \Omega(\Theta_i, \phi)$, where $\Omega(\Theta_i, \phi)$ is the set of solutions of the *equilibrium problem*:

$$\Theta_i(x^*, y) + \phi(y) - \phi(x^*) \geqslant 0, \quad \forall y \in C.$$

In particular, If $\phi = 0$ and $N = 1$, then the problem (1.1) is reduced to the *equilibrium problem* (EP).

* Corresponding author.
*E-mail addresses:* changss@yahoo.cn (S.-s. Chang), machanck@inet.polyu.edu.hk (C.K. Chan), majlee@inet.polyu.edu.hk (H.W.J. Lee), yanglizxs@yahoo.com.cn (L. Yang).

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It is well-known that the SMEP includes fixed point problem, optimization problem, variational inequality problem, and Nash equilibrium problem as its special cases.

Let $C$ be a closed convex subset of a Hilbert space $H$. A family of mappings $\mathscr{S} := \{S(s) : 0 \leqslant s < \infty\} : C \to C$ is said to be a *nonexpansive semi-group*, if it satisfies the following conditions:

(i) $S(s+t) = S(s)S(t),\ \forall s,t \in \mathscr{R}^+$ and $S(0) = I$;
(ii) $\|S(s)x - S(s)y\| \leqslant \|x-y\|,\ \forall x,y \in C,\ s \geqslant 0$;
(iii) the mapping $t \mapsto S(t)x$ is continuous for each $x \in C$.

We denote by $F(\mathscr{S})$ the set of common fixed points of $\mathscr{S} = \{S(s) : s \geqslant 0\}$, i.e., $F(\mathscr{S}) = \bigcap_{s\geqslant 0} F(S(s))$

In 2008, Ceng-Yao [1] used the well-known KKM technique to prove that the sequence generated by the hybrid iterative scheme converges to a common element of the set of solutions of MEP and the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space.

Very recently, Saeidi [2] introduced an iterative algorithm for finding a common element of the set of solutions for a system of equilibrium problems and the set of common fixed points for a finite family of nonexpansive mappings and the set of common fixed points for a left amenable nonexpansive semi-group in a Hilbert space.

Motivated and inspired by Ceng-Yao [1] and Saeidi [2], the purpose of this paper is to introduce an iterative algorithm for finding a common element of the set of solutions for a system of mixed equilibrium problems and the set of common fixed points for an infinite family of strictly pseudo-contractive mappings and the set of common fixed points for nonexpansive semi-groups in Hilbert space. Under suitable conditions some strong convergence theorems to converging to the unique common element are proved. The results presented in the paper improve and extend the corresponding results in [1–4].

## 2. Preliminaries

Let $H$ be a real Hilbert space and $C$ be a nonempty closed convex subset of $H$. Then for each $x \in H$, there exists a unique nearest point $u \in C$ such that

$$\|x-u\| \leqslant \|x-y\|, \quad \forall y \in C.$$

The mapping $P_C : x \to u$ is called *metric projection of H onto C*. It is known that $P_C$ is nonexpansive and for $x \in H$ and $u \in C$

$$u = P_C(x) \Longleftrightarrow \langle x-u,\ u-y\rangle \geqslant 0, \quad \forall y \in C.$$

Recall that a Banach space $E$ is said to satisfy *the Opial condition*, if for any sequence $\{x_n\}$ in $E$ with $x_n \rightharpoonup x$, then for every $y \in E$ with $y \neq x$ we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|.$$

It is well-known that each Hilbert space satisfies the Opial condition.

Recall that a mapping $f : H \to H$ is said to be contractive, if there exists a constant $\xi \in (0,1)$ such that

$$\|f(x) - f(y)\| \leqslant \xi\|x-y\|, \quad \forall x,y \in H.$$

A mapping $V : C \to H$ is said to be *k-strictly pseudo-contractive*, if there exists a constant $k \in [0,1)$ such that

$$\|Vx - Vy\|^2 \leqslant \|x-y\|^2 + k\|(I-V)x - (I-V)y\|^2, \quad \forall x,y \in C.$$

**Lemma 2.1** [5]. *Let $V : C \to H$ be a k-strict pseudo-contraction, then*

(1) *the fixed point set $F(V)$ of $V$ is closed convex so that the projection $P_{F(V)}$ is well defined;*
(2) *define a mapping $T : C \to H$ by*

$$Tx = \gamma x + (1-\gamma)Vx, \quad x \in C. \tag{2.1}$$

*If $\gamma \in [k,1)$, then $T$ is a nonexpansive mapping such that $F(V) = F(T)$.*

**Definition 2.1.** A family of mappings $\{V_i : C \to H\}_{i=1}^{\infty}$ is called *a family of uniformly k-strict pseudo-contractions*, if there exists a constant $k \in [0,1)$ such that

$$\|V_i x - V_i y\|^2 \leqslant \|x-y\|^2 + k\|(I-V_i)x - (I-V_i)y\|^2, \quad \forall x,y \in C,\ \forall i \geqslant 1.$$

**Definition 2.2.** Let $\{V_i : C \to C\}$ be a countable family of uniformly $k$-strict pseudo-contractions. Let $\{T_i : C \to C\}_{i=1}^{\infty}$ be the sequence of nonexpansive mappings defined by (2.1), i.e.,

$$T_i x = \gamma x + (1-\gamma)V_i x, \quad x \in C,\ \forall i \geqslant 1 \text{ with } \gamma \in [k,1). \tag{2.2}$$