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Common fixed points for four maps in cone metric spaces

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ABSTRACT

The existence of coincidence points and common fixed points for four mappings satisfying generalized contractive conditions without exploiting the notion of continuity of any map involved therein, in a cone metric space is proved. These results extend, unify and generalize several well known comparable results in the existing literature.

1. Introduction and preliminaries

Jungck [9] defined a pair of self-mappings to be weakly compatible if they commute at their coincidence points. In recent years, several authors have obtained coincidence point results for various classes of mappings on a metric space, utilizing these concepts. For a survey of coincidence point theory, its applications, comparison of different contractive conditions and related results, we refer to [4,6,11] and the references contained therein. Huang and Zhang [5] generalized the concept of a metric space, replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions. Subsequently, Abbas and Jungck [2] and Abbas and Rhoades [1] studied common fixed point theorems in cone metric spaces (see also, [3,5,8,13] and the references mentioned therein). In this paper, common fixed point theorems for two pairs of weakly compatible maps, which are more general than R-weakly commuting and compatible mappings, are obtained in the setting of cone metric spaces, without exploiting the notion of continuity. It is worth mentioning that our results do not require the assumption that the cone is normal. Our results extend and unify various comparable results in the literature [2–4,7,8]. Consistent with Huang and Zhang [5], the following definitions and results will be needed in the sequel.

Let E be a real Banach space. A subset P of E is called a *cone* if and only if:

(a) P is closed, non-empty and $P \neq \{0\}$;
(b) $a, b \in R,\ a, b \geqslant 0,\ x, y \in P$ imply that $ax + by \in P$;
(c) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leqslant$ with respect to P by $x \leqslant y$ if and only if $y - x \in P$, where $x \ll y$ means that $y - x \in \operatorname{int} P$ (interior of P). A cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leqslant x \leqslant y \text{ implies } \|x\| \leqslant K\|y\|.$$

The least positive number satisfying the above inequality is called the *normal constant* of P.

Rezapour and Hamlbarani [12] proved that there is no normal cone with normal constant $K < 1$ and for each $k > 1$ there are cones with normal constants $K > k$.

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Definition 1.1. Let X be a non-empty set. Suppose that the mapping $d: X \times X \to E$ satisfies:

(d1) $0 \leqslant d(x,y)$ for all $x, y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
(d2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(d3) $d(x,y) \leqslant d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X,d) is called a *cone metric space*. The concept of a cone metric space is more general than that of a metric space.

Definition 1.2. Let (X,d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. For every $c \in E$ with $0 \ll c$, we say that $\{x_n\}$ is

(i) a *Cauchy* sequence if there is an N such that, for all $n, m > N$, $d(x_n, x_m) \ll c$.
(ii) a *convergent* sequence if there is an N such that, for all $n > N$, $d(x_n, x) \ll c$ for some x in X.

A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X. It is known that $\{x_n\}$ converges to $x \in X$ if and only if $d(x_n, x) \to 0$ as $n \to \infty$. A subset A of X is closed if every Cauchy sequence in A has its limit point in A.

Definition 1.3. Let f and g be self-maps on a set X. If $w = fx = gx$, for some x in X, then x is called coincidence point of f and g, where w is called a point of coincidence of f and g.

Definition 1.4. Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at every coincidence point.

Remark 1.5. If E is a real Banach space with a cone $P$and if $a \leqslant ha$ where $a \in P$ and $h \in (0,1)$, then $a = 0$.

Remark 1.6. If $0 \leqslant u \ll c$ for each $0 \ll c$ then $u = 0$.

2. Common fixed point results

The following Lemma not only improves but also extends Lemma 1 of [10] cone metric spaces.

Lemma 2.1. *Let f, g, S and T be self-maps on a cone metric space X with cone P having non-empty interior, satisfying $f(X) \subset T(X)$ and $g(X) \subset S(X)$. Define $\{x_n\}$ and $\{y_n\}$ by $y_{2n+1} = fx_{2n} = Tx_{2n+1}$, $y_{2n+2} = gx_{2n+1} = Sx_{2n+2}$, $n \geqslant 0$. Suppose that there exist a $\lambda \in [0,1)$ such that*

$$d(y_n, y_{n+1}) \leqslant \lambda d(y_{n-1}, y_n) \quad \text{for each } n \geqslant 1. \tag{2.1}$$

Then either

(a) $\{f,S\}$ and $\{g,T\}$ have coincidence points, and $\{y_n\}$ converges, or
(b) $\{y_n\}$ is Cauchy.

Moreover, if X is complete, then $\{y_n\}$ converges to a point $z \in X$ and

$$d(y_n, z) \leqslant \frac{\lambda^n}{1-\lambda} d(y_0, y_1) \quad \text{for each } n > 0. \tag{2.2}$$

Proof. To prove part (a), suppose that there exists an n such that $y_{2n} = y_{2n+1}$. Then, from the definition of $\{y_n\}$, $gx_{2n-1} = Sx_{2n} = fx_{2n} = Tx_{2n+1}$, and f and S have a coincidence point. Moreover, from (2.1),

$$d(y_{2n+1}, y_{2n+2}) \leqslant \lambda d(y_{2n}, y_{2n+1}) = 0,$$

so that $y_{2n+1} = y_{2n+2}$; i.e., $fx_{2n} = Tx_{2n+1} = gx_{2n+1} = Sx_{2n+2}$, and g and T have a coincidence point. In addition, repeated use of (2.1) yields $y_{2n} = y_m$ for each $m > 2n$, and hence $\{y_n\}$ converges.

The same conclusion holds if $y_{2n+1} = y_{2n+2}$ for some n.

For part (b), assume that $y_{2n} \neq y_{2n+1}$ for all n. Then (2.1) implies that

$$d(y_n, y_{n+1}) \leqslant \lambda^n d(y_0, y_1).$$

For any $m, n \in N$ with $m > n$ it follows that