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Certain subclasses of multivalent analytic functions defined by multiplier transforms

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ABSTRACT

By making use of the principle of subordination between analytic functions and a family of multiplier transforms, we introduce and investigate some new subclasses of multivalent analytic functions. Such results as inclusion relationships, subordination and superordination properties, integral-preserving properties, argument estimates and convolution properties are proved.

1. Introduction

Let $\mathcal{A}_p(n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \quad (p,\ n \in \mathbb{N} := \{1,2,3,\ldots\}), \tag{1.1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z :\ z \in \mathbb{C} \text{ and } |z| < 1\}.$$

Let $\mathcal{H}[a, n]$ be the class of analytic functions of the form

$$h(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (z \in \mathbb{U}).$$

Let $f,\ g \in \mathcal{A}_p(n)$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=n}^{\infty} b_{p+k} z^{p+k}.$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=n}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g * f)(z).$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k \quad (n \in \mathbb{N}),$$

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which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$\Re(p(z)) > 0 \quad (z \in \mathbb{U}).$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \to f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In a recent paper, Cătaş [3] defined a class of multiplier transforms $\mathcal{I}_{p,n}(m, \lambda, l)$ on $\mathcal{A}_p(n)$ by the following infinite series:

$$\mathcal{I}_{p,n}(m,\lambda,l)f(z) := z^p + \sum_{k=n}^{\infty} \left(\frac{p+\lambda k+l}{p+l}\right)^m a_{p+k} z^{p+k} \tag{1.2}$$
$$(z \in \mathbb{U};\ p,\ n \in \mathbb{N};\ m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \lambda,\ l \geqq 0).$$

It is easily verified from (1.2) that

$$\lambda z(\mathcal{I}_{p,n}(m,\lambda,l)f)'(z) = (p+l)\mathcal{I}_{p,n}(m+1,\lambda,l)f(z) - [(1-\lambda)+l]\mathcal{I}_{p,n}(m,\lambda,l)f(z). \tag{1.3}$$

It should be remarked that the class of multiplier transforms $\mathcal{I}_{p,n}(m, \lambda, l)$ is a generalization of several other linear operators considered in earlier investigations (see [2,4,6–8,14]).

More recently, Acu et al. [1] discussed the radius of starlikeness of a subclass of p-valent functions defined by $\mathcal{I}_{p,n}(m, \lambda, l)$, Cătaş et al. [5] obtained some subordination results associated with $\mathcal{I}_{p,n}(m, \lambda, l)$. In this paper, by making use of the multiplier transforms $\mathcal{I}_{p,n}(m, \lambda, l)$ and the above-mentioned principle of subordination between analytic functions, we introduce and investigate the following subclasses of the class $\mathcal{A}_p(n)$ of p-valent analytic functions.

Definition 1. A function $f \in \mathcal{A}_p(n)$ is said to be in the class $\mathcal{R}_{p,n}(m, \lambda, l;\ \alpha;\ \phi)$ if it satisfies the subordination condition

$$\frac{1}{p-\alpha}\left(\frac{z(\mathcal{I}_{p,n}(m,\lambda,l)f)'(z)}{\mathcal{I}_{p,n}(m,\lambda,l)f(z)} - \alpha\right) \prec \phi(z) \quad (z \in \mathbb{U};\ \phi \in \mathcal{P}), \tag{1.4}$$

where (and throughout this paper unless otherwise mentioned) the parameters α, p, n, m, λ and l are constrained as follows:

$$0 \leqq \alpha < p;\ p,\ n \in \mathbb{N};\ m \in \mathbb{N}_0 \quad \text{and} \quad \lambda,\ l \geqq 0. \tag{1.5}$$

For convenience, we write

$$\mathcal{R}_{p,n}\left(m,\lambda,l;\alpha;\frac{1+Az}{1+Bz}\right) =: \mathcal{R}_{p,n}(m,\lambda,l;\alpha;A,B) \quad (-1 \leqq B < A \leqq 1).$$

Definition 2. A function $f \in \mathcal{A}_p(n)$ is said to be in the class $\mathcal{K}_{p,n}(m, \lambda, l;\ \beta;\ \phi)$ if it satisfies the subordination condition

$$(1-\beta)\frac{\mathcal{I}_{p,n}(m,\lambda,l)f(z)}{z^p} + \beta\frac{\mathcal{I}_{p,n}(m+1,\lambda,l)f(z)}{z^p} \prec \phi(z) \quad (z \in \mathbb{U};\ \beta \in \mathbb{C};\ \phi \in \mathcal{P}). \tag{1.6}$$

In the present paper, we aim at proving such results as inclusion relationships, subordination and superordination properties, integral-preserving properties, argument estimates and convolution properties for the classes $\mathcal{R}_{p,n}(m, \lambda, l;\ \alpha;\ \phi)$ and $\mathcal{K}_{p,n}(m, \lambda, l;\ \beta;\ \phi)$.

2. Preliminary results

In order to establish our main results, we need the following lemmas.

Lemma 1 (See [9]). *Let $\vartheta,\ \gamma \in \mathbb{C}$. Suppose that φ is convex and univalent in $\mathbb{U}$ with*

$$\varphi(0) = 1 \quad \textit{and} \quad \Re(\vartheta\varphi(z) + \gamma) > 0 \quad (z \in \mathbb{U}).$$