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Stochastic evolution of 2D crystals

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ABSTRACT

We study a crystalline version of the modified Stefan problem in the plane. The feature of our approach is that, we consider a model with stochastic perturbations and assume the interfacial curve to be a polygon. The existence of solution to our stochastic system is established. Galerkin's method is one of the main tools used in the proof of our assertion.

1. Introduction and notations

In our paper, we study a crystalline version of the modified Stefan problem with a stochastic source term. We assume that the interfacial curve is a polygon. Let us stress in this place that admitting non-smooth interfaces is natural in modeling crystal evolution.

Our goal is to show the existence of solution to the crystalline version of the stochastic free boundary problem with a singular free boundary. The feature of our approach is that, we consider the stochastic heat equation and assume the interfacial curve to be a polygon, which is a non-smooth interface.

The Stefan type problems have been intensively studied in recent years, but there are few publications, which consider this subject with stochastic perturbations. One of the notable exceptions is the paper of Yip [21], where the mathematical existence for dendritic crystal growth in a model that incorporates stochastic perturbations has been proved. To put it more precisely, the author introduces thermal fluctuations into a model of crystal growth and shows an existence result for an evolution process of crystal shape and heat distribution satisfying the Gibbs–Thomson condition and the stochastic heat equation. Some other articles, which are also worth mentioning in the context of the Stefan type problems under a random disturbance are the papers of Ishikawa and Miyajima [12] and Barbu and Da Prato [3]. In [12], the authors study the influence of the random fluctuation of the temperature on the crystal growth in the Stefan problem. They propose the stochastic phase field model and use it to analyze the mentioned crystal growth processes. In [3], the existence for the two-phase Stefan problem perturbed by a stochastic noise is considered. We shall pursue the direction from the cited papers.

The description of the problem of melting and growing a crystal requires setting this problem into the frame of two-phase thermodynamics analysis. It has been done by Gurtin [8,9] and Angenent and Gurtin [2]. The theory exposed there is based on three fundamental laws, namely the laws of: the balance of capillary forces, the balance of energy and the growth of entropy. In this treatment, the interface is endowed with a thermodynamical structure. In its simplest case, we obtain the classical Stefan problem, i.e., the change of phase occurs at constant temperature. The thermodynamical theory of Gurtin and Angenent has been extended by Gurtin and Matias [11] to crystalline motion, i.e., the evolution of polygonal interfaces. In our case, the equations of stochastic heat transport and motion of the free boundary:

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$$u_t = \Delta u + f\frac{dW}{dt} \quad \text{in} \quad \bigcup_{t\in(0,T)} (D_1(t) \cup D_2(t)), \quad u(0) = u_0, \tag{1}$$

$$V_i = -[[\nabla u]]v_i, \quad i = 1, \ldots, N, \tag{2}$$

are supplemented with the following version of the averaged Gibbs–Thomson condition that the temperature u must satisfy on the interface

$$\int_\Omega \int_{s_i(t)} u \; dl \; dP = \Gamma_i - \beta_i L_i(t) V_i(t) \quad \text{for} \quad i = 1, \ldots, N. \tag{3}$$

The kinetic coefficients $\beta_i > 0, \; i = 1, \ldots, N$, are constants and so are the quantities $\Gamma_i, \; i = 1, \ldots, N$. The Γ_i's are defined as follows

$$\Gamma_i = \begin{cases} -\ell_i, & \text{if } s \text{ is locally convex near both vertices } r_i, r_{i+1}, \\ \ell_i, & \text{if } s \text{ is locally concave near both vertices } r_i, r_{i+1}, \\ 0, & \text{otherwise,} \end{cases}$$

where ℓ_i stands for the length of the edge of the Wulff shape with normal v_i. The ratio Γ_i/L_i may be related to the curvature of s_i . We use the definition of curvature, which does not take into account the differential structure of s. For more details see [8,9,19].

In order to close the above system, we impose the Dirichlet boundary data

$$u_{|\partial D} = 0 \quad \text{for} \quad t \geqslant 0. \tag{4}$$

We shall deal with the model (1)–(4). Our aim is to show the existence of weak solution to this stochastic system. As we have already mentioned at the beginning of the current Section, with a few exceptions, there are not many works devoted to the Stefan type problems with a stochastic source term. We partly try to fill this gap and consider the modified Stefan problem that incorporates stochastic perturbations.

The notations used in (1)–(4), as well as some other terminologies, notations and details involved in the considered model are as follows.

We denote by $u(t)$ the normalized temperature and by $D, D_1(t), D_2(t)$ the bounded regions of the plane, such that $\overline{D_1}(t) \subset D$ and $\overline{D} = \overline{D_1}(t) \cup \overline{D_2}(t)$. We assume that a crystal occupies the region $D_1(t)$ of the container D and that the remaining part $D_2(t)$ of D is filled with melt. We also suppose that the boundary ∂D of D is smooth and that the interface $s(t) = \partial D_1(t) \cap \partial D_2(t)$ is a polygon with facets $s_i(t), \; i = 1, \ldots, N, \; s(t) = \bigcup_{i=1}^N s_i(t)$. In addition, we assume that: the i-th facet $s_i(t)$ of $s(t)$ is determined by its vertices $r_i(t)$ and $r_{i+1}(t), \; L_i(t) = |r_i(t) - r_{i+1}(t)|$ is the length of $s_i(t)$, and $V_i(t)$ denotes the velocity of $s_i(t)$ in the direction of the outer normal v_i to $\partial D_1(t)$ (wherever it is defined) perpendicular to the side $s_i(t)$. The latest means that

$$V_i(t) = \frac{d}{dt} z_i(t),$$

where

$$z_i(t) = \begin{cases} \operatorname{dist}(l_i(t), l_i(0)), & \text{if } (r_i(t) - r_i(0)) \cdot v_i > 0, \\ -\operatorname{dist}(l_i(t), l_i(0)), & \text{if } (r_i(t) - r_i(0)) \cdot v_i \leqslant 0, \end{cases}$$

and $l_i(t), \; l_i(0)$ are the lines containing $s_i(t), \; s_i(0)$, respectively.

We also assume that the polygonal interfaces are admissible. By admissibility, we mean here that the outer normals v_i to the facets s_i belong to the set of normals of a given Wulff shape (see [10]).

On the other hand, we make the following assumptions concerning the process W in (1).

Let $(\Omega, \mathscr{F}, P, \{\mathscr{F}_t\})$ be a probabilized stochastic basis, i.e., $(\Omega, \mathscr{F}, P)$ is a probability space and $\{\mathscr{F}_t\} \subset \mathscr{F}, \; t \in [0, T]$, stands for the filtration. Then, we assume that W in (1) is the real Wiener process on $(\Omega, \mathscr{F}, P)$, satisfying the conditions:

$$W(t) \text{ is } \mathscr{F}_t - \text{measurable},$$

$$W(t) - W(s) \text{ is independent of the } \sigma - \text{field } \mathscr{F}_s, \text{ for any } s \leqslant t.$$

Furthermore, f in (1) is the $\mathscr{F}_t$ – adaptable stochastic process from appropriate space function.

Finally, observe that the definition of the velocity V_i in (2) involves the jump $[[\cdot]]$ across the polygon $s(t)$. It means that $[[\cdot]]$ in (2) is given by

$$[[\phi]](x_0) = \lim_{D_2(t) \ni x \to x_0} \phi(x) - \lim_{D_1(t) \ni x \to x_0} \phi(x), \quad x_0 \in \partial D_1(t) \cap \partial D_2(t).$$

Let us now give some additional comments and remarks on the published mathematical literature concerning the topic.

We assume in (1)–(4) that the interfacial curve is a polygon, which is non-smooth interface. It is worth mentioning in this place that the problem in (1)–(4) without stochastic perturbations has already been investigated mathematically (see: [16–20,5]). On the other hand, the modified Stefan problem for smooth interfaces has already been widely studied in recent two