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When is the numerical range of a nilpotent matrix circular?

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ABSTRACT

The problem formulated in the title is investigated. The case of nilpotent matrices of size at most 4 allows a unitary treatment. The numerical range of a nilpotent matrix M of size at most 4 is circular if and only if the traces $\operatorname{tr} M^*M^2$ and $\operatorname{tr} M^*M^3$ are null. The situation becomes more complicated as soon as the size is 5. The conditions under which a 5×5 nilpotent matrix has circular numerical range are thoroughly discussed.

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1. Introduction

A nilpotent operator is a linear operator T with the property $T^n = 0$ for some positive integer n. We say n is the *order* of T if n is the least positive integer with the property $T^n = 0$.

Recall that the numerical range $W(T)$ of a bounded operator T acting on some complex Hilbert space is the image of the unit sphere of that space under the quadratic form associated to T, that is $W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}$. One of the best known properties of numerical ranges is the so called Toeplitz–Hausdorff theorem (see [4, Ch. 22]), saying that numerical ranges are convex subsets of the complex plane. The supremum $w(T)$ of the absolute values of the complex numbers in $W(T)$ is called the numerical radius of T. The basic facts of the theory of numerical ranges are contained in [3] and [4].

There are several extensions of the notion of numerical range. Some are interesting both in their own right and for the sake of their applications (for instance to quantum information theory, [9]). The numerical ranges of nilpotent operators are insufficiently known as one can deduce from the recent paper [5] where the author uses dilation theory arguments to state and prove that Hilbert space nilpotent operators always have circular numerical ranges. That proof is faulty since even 3×3 nilpotent matrices can have non-circular numerical ranges (see [6] for examples of such matrices). Thus it seems both interesting and useful investigating when a $n \times n$ nilpotent matrix with complex entries has circular numerical range. Here are the first remarks on that issue.

Remark 1. A 2×2 nilpotent matrix M always has circular numerical range. The disk $W(M)$ is centered at the origin and its radius can be calculated with the formula $w(M) = \sqrt{\operatorname{tr}(M^*M)}/2$.

Indeed, one of the most popular theorems on numerical ranges, the *elliptic range theorem*, says that the numerical range of a 2×2 matrix with complex entries is an elliptical disk (possibly degenerate, that is possibly reduced to its focal axis), whose foci are the eigenvalues of that matrix. So, if the matrix is nilpotent, those eigenvalues equal 0, hence the numerical range is a circular disk centered at the origin (possibly reduced to its center, if we consider the null matrix). The formulae for the semi-axes of the elliptical disk (see [3] or [4]), lead to the equality $w(M) = \sqrt{\operatorname{tr}(M^*M)}/2$.

The situation when 3×3 nilpotent matrices have circular numerical ranges is completely described in [6, Theorem 4.1], according to which:

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Remark 2. A 3×3 nilpotent matrix M has circular numerical range if and only if

$$\operatorname{tr}(M^*M^2) = 0, \tag{1}$$

in which case, the numerical radius is computable with the formula

$$w(M) = \frac{\sqrt{\operatorname{tr}(M^*M)}}{2}.$$

The case of 4×4 matrices is covered by [2]. Indeed, a consequence of [2, Corollaries 5 and 6] is the following:

Remark 3. A 4×4 nilpotent matrix M has circular numerical range if and only if

$$\operatorname{tr}(M^*M^2) = 0 \quad \text{and} \quad \operatorname{tr}(M^*M^3) = 0, \tag{2}$$

in which case, the numerical radius is computable with the formula

$$w(M) = \sqrt{\frac{\operatorname{tr}(M^*M) + \sqrt{(\operatorname{tr}(M^*M))^2 - 64\det(\Re(M))}}{8}},$$

where $\Re(M) = (1/2)(M + M^*)$ denotes the real part of M. In all the cases above, the circular disks are centered at the origin. Hence:

Remark 4. A nilpotent $k \times k$ matrix of size $2 \leqslant k \leqslant 4$ has circular numerical range if and only if

$$\operatorname{tr}(M^*M^n) = 0, \quad n = 2, 3, 4, 5, \ldots. \tag{3}$$

If the size of a nilpotent matrix M is k, then clearly $M^n = 0$ if $n \geqslant k$, so the only interesting values of n in condition (3) are those between 2 and k. The first reaction is to call (3) the *trace condition* and try to prove that it is the if and only if characterization of the situation when a nilpotent matrix with complex entries has circular numerical range. As we show in the next section, this is not true, and counterexamples can be produced using 5×5 matrices. After considering such counterexamples, we completely characterize the situation when a nilpotent 5×5 matrix has circular numerical range. This (Theorem 2) is the main result of the current paper. After proving it, we discuss thoroughly when a 5×5 nilpotent matrix satisfying, respectively non-satisfying, the trace condition (3) has circular numerical range.

2. The main results

We begin by noting that the trace condition is not necessary for the circularity of the numerical range of a 5×5 matrix.

Example 1. The block–diagonal matrix

$$M = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix},$$

where

$$B_1 = \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has circular numerical range if $|a|$ is large enough, but $\operatorname{tr}(M^*M^2) = 1 \neq 0$.

If $|a|$ is large enough, then $W(B_2) \subseteq W(B_1)$ since $W(B_2)$ is bounded and $W(B_1)$ is the circular disk centered at the origin and having radius $|a|/2$ (by Remark 1). In that case, one has that $W(M) = W(B_1)$ because $W(M)$ is the convex hull of the union $W(B_1) \cup W(B_2)$ (a fact valid for any block–diagonal matrix).

In the case of 5×5 nilpotent matrices, the trace condition is not sufficient for the circularity of the numerical range. Indeed:

Example 2. The matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{4}$$

satisfies the trace condition, but its numerical range exhibits a flatness on the boundary (see Fig. 1).

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