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The method of Kantorovich majorants to nonlinear singular integral equation with shift

S.M. Amer *, S. Dardery

Math. Dept., Faculty of Science, Zagazig University, Zagazig, Egypt

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ABSTRACT

The paper is concerned with the applicability of the method of Kantorovich majorants to nonlinear singular integral equation with shift. The result is illustrated in the generalized Hölder space.

0. Introduction

The classical theory of singular integral equations (SIE) is rather complete (see [9,10,19,20,22,24], and others). The theory of approximation methods and its applications for the solution of linear and nonlinear singular integral equations (LSIE) and (NSIE) has been developed by many authors [3,6,11,12,17]. There is a literature on the successful development of the nonlinear singular integral equations with shift (NSIES) [1,4,5,15,18,21]. The Noether theory of singular integral operators with shift (SIOS) is developed for a closed and open contour ([2,10,13,14,16,18] and others). The theory of singular integral equations with shift (SIES) is an important part of integral equations because of its recent applications in many fields of physics and engineering [8,14,16].

In this paper the method of Kantorovich majorants has been applied to the following NSIES:

$$(Q(u))(t) = a(t)u(t) + b(t)u(\alpha(t)) + \frac{c(t)}{\pi i}\int_L \frac{u(\tau)}{\tau - t}d\tau + \frac{d(t)}{\pi i}\int_L \frac{u(\tau)}{\tau - \alpha(t)}d\tau - \frac{1}{\pi i}\int_L \left\{\frac{\psi_1(\tau, u(\tau))}{\tau - t} + \frac{\psi_2(\tau, u(\tau))}{\tau - \alpha(t)}\right\}d\tau = 0, \quad \textit{for all} \quad t \in L, \tag{0.1}$$

under the following conditions

$$a(t) = \psi_{1u}(t, u_0(t)) - c(t), \quad b(t) = d(t) - \psi_{2u}(\alpha(t), \quad u_0(\alpha(t))), \tag{0.2}$$

for initial value u_0, in the generalized Hölder space $H_{\varphi,m}(L)$, where L is a simple smooth closed Lyapunov contour, dividing the complex plane into two domains D^+ (the interior domain) and D^- (the exterior domain), $D = D^+ \cup D^-$, and the homeomorphism $\alpha : L \to L$ is the preserving orientation and satisfies the Carleman condition:

$$\alpha(\alpha(t)) = \alpha_2(t) = t; \quad t \in L, \tag{0.3}$$

and the derivative $\alpha'(t) \neq 0$ satisfies the usual Hölder condition.

The functions $a(t), b(t), c(t)$ and $d(t)$ belong to the generalized Hölder space $H_{\varphi,m}(L)$. Moreover, the functions $\psi_1(t, u(t))$ and $\psi_2(t, u(t))$ have partial derivatives up to $(m-1)$-order, and satisfy the following conditions:

* Corresponding author.
E-mail address: amrsammer@hotmail.com (S.M. Amer).

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$$\left|\frac{\partial^l \psi_1(t_1,u_1)}{\partial t^i \partial u^j} - \frac{\partial^l \psi_1(t_2,u_2)}{\partial t^i \partial u^j}\right| \leqslant c_l(r)\{\varphi(|t_1-t_2|) + |u_1-u_2|\} \tag{0.4}$$

$$\left|\frac{\partial^l \psi_2(t_1,u_1)}{\partial t^i \partial u^j} - \frac{\partial^l \psi_2(t_2,u_2)}{\partial t^i \partial u^j}\right| \leqslant d_l(r)\{\varphi_1(|t_1-t_2|) + |u_1-u_2|\} \tag{0.5}$$

for $|u_i - u_0| < r, i = 1,2$, where $\varphi, \varphi_1 \in \Phi, i+j = l, l = 0,1,\ldots,m-1$ and $c_l(r), d_l(r)$ are positive increasing functions the functions $\psi_1(t,u(t))$ and $\psi_2(t,u(t))$ belong to the space $H_{\varphi,m}(L)$ for any $u \in H_{\varphi,m}(L)$, [23].

The Eq. (0.1) under condition (0.2) has been studied by modified Newton–Kantorovich method in [5].

1. Formulation of the problem

Let $f : \overline{B}(u_0,R) \subset X \to Y$ is a nonlinear operator defined on the closure of a ball $B(u_0,R) = \{u : u \in X, \|u-u_0\| < R\}$ in a Banach space X into a Banach space Y. We give new conditions to ensure the convergence of Newton–Kantorovich approximations toward a solution of $f(u) = 0$, under the hypothesis that f is Frechet differentiable in $B(u_0,R)$, and that its derivative f' satisfies the local Lipschitz condition

$$\|f'(u_1) - f'(u_2)\| \leqslant k(r)\|u_1-u_2\|, \quad u_1,u_2 \in \overline{B}(u_0,r); \quad 0 < r < R, \tag{1.1}$$

where $k(r)$ is non-decreasing function on the interval $[0,R]$ and

$$k(r) = \sup\left\{\frac{\|f'(u_1)-f'(u_2)\|}{\|u_1-u_2\|} : u_1,u_2 \in \overline{B}(u_0,r); u_1 \neq u_2\right\}. \tag{1.2}$$

Define a scalar function $\phi : [0,R] \to [0,\infty)$ by

$$\phi(r) = \varepsilon + \mu\int_0^r \omega(t)dt - r, \tag{1.3}$$

where the function

$$\omega(r) = \int_0^r k(t)dt, \tag{1.4}$$

and

$$\varepsilon = \|f'(u_0)^{-1}f(u_0)\|, \quad \mu = \|f'(u_0)^{-1}\|. \tag{1.5}$$

Theorem 1.1 [25]. *Suppose that the function ϕ has a unique positive root r_* in $[0,R]$ and $\phi(R) \leqslant 0$. Then the equation $f(u) = 0$ has a unique solution u_* in $B(u_0,R)$ and the Newton–Kantorovich approximations*

$$u_n = u_{n-1} - f'(u_{n-1})^{-1}f(u_{n-1}), \quad n \in N, \tag{1.6}$$

are defined for all $n \in N$, belong to $B(u_0,r_*)$ and converge to u_*. Moreover, the following estimate holds

$$\|u_{n+1}-u_n\| \leqslant r_{n+1} - r_n, \quad \|u_* - u_n\| \leqslant r_* - r_n, \tag{1.7}$$

where the sequence $(r_n)_{n\in N}$ increasing and convergent to r_*, is defined by the recurrence formula

$$r_0 = 0, \quad r_{n+1} = r_n - \frac{\phi(r_n)}{\phi'(r_n)}, \quad n \in N. \tag{1.8}$$

Our aim is to investigate sufficient conditions, which ensure that the NSIES (0.1) verifies the hypotheses of Theorem 1.1.

2. Some basic results

Definition 2.1 (*[12,19]*). We denote by $c(L)$ the space of all continuous functions $u(t)$ defined on L with the norm:

$$\|u\|_{c(L)} = \max_{t\in L}|u(t)|. \tag{2.1}$$

Definition 2.2 (*[10,23]*). For the continuous function $u(t)$ defined on a closed interval $[a,b]$, the function $\omega_u^m(\delta) = \omega^m(u,\delta)$ is called the modulus of continuity of order m of the function $u(t)$, and defined on a closed interval $\left[0, \frac{b-a}{m}\right]$ as follows

$$\omega_u^m(\delta) = \sup_{0\leqslant h\leqslant\delta;\delta>0}|\Delta_h^m(u;x)|, \tag{2.2}$$