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Analytic transient solutions of a cylindrical heat equation with a heat source

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ABSTRACT

The method of separation of variables is applied in order to investigate the analytical solutions of a certain two-dimensional cylindrical heat equation. In the analysis presented here, the partial differential equation is directly transformed into ordinary differential equations. The closed-form transient temperature distributions and heat transfer rates are generalized for a linear combination of the products of Fourier–Bessel series of the exponential type. Relevant connections with some other closely-related recent works are also indicated.

1. Introduction

A systematic procedure for determining the separation of variables for a given partial differential equation can be found in such oft-cited works as [1,2]. Analytical solutions are particularly important and useful. However, a heat conduction problem involving two dimensions and fairly general boundary conditions of the type considered in our present study is presumably not solved in the existing literature on this subject. The method of separation of variables is applied here in order to investigate analytical solutions of the general heat conduction problem proposed in this study. The partial differential equations are transformed into ordinary differential equations by separating the independent variables involved in the problem. The temperature distribution of fins under transient condition are important for proper prediction and control of the fin performance. Closed-form analytical solution for the transient temperature distribution and the heat transfer rate would provide continuous physical insight which is much better than discrete numbers from a viewpoint of numerical computation. The main purpose of this study is to investigate the analytical transient solutions by using the method of separation of variables.

2. Formulation of the problem

A number of restrictive assumptions are introduced before studying the transient analysis, some of which are due to Kung and Srivastava [3]. One can apply Fourier's law and Newton's energy conservation law to form the two-dimensional heat equation, together with the initial condition and the boundary conditions as follows:

$$\frac{\partial u(r,z,t)}{\partial t} = \frac{\partial^2 u(r,z,t)}{\partial z^2} + \frac{\partial^2 u(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z,t)}{\partial r} + \delta(r,z,t). \tag{1}$$

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Initially, the cylindrical fin is in equilibrium with the surrounding fluid, that is,

$$t = 0: \quad u(r, z, 0) = 0. \tag{2}$$

The boundary conditions are given by

$$t > 0 \quad \text{and} \quad r = 0: \quad u(0, z, t) = \text{finite}; \tag{3}$$
$$r = 1: \quad u_r(1, z, t) = 0; \tag{4}$$
$$z = 0: \quad -u_z(r, 0, t) + \text{Bi} \cdot u(r, 0, t) = \text{Bi} + q; \tag{5}$$
$$z = \ell: \quad u_z(r, \ell, t) + \text{Bi}_\ell \cdot u(r, \ell, t) = 0, \tag{6}$$

where Bi and Bi_ℓ are the Biot numbers on the root surface and the lateral surface, respectively, and q is the constant heat flux at the root of the fin.

3. Main results and their consequences

The method of superposition and separation of variables is a very classical method to solve a given initial- and boundary-value problem for partial differential equations (see, for example, [1,2]). However, the partial differential Eq. (1) when directly transformed into ordinary differential equations by the method of separation of variables method does not seem to have been considered in the existing literature on this subject.

One can first decompose the Eq. (1) into the homogeneous problem, and solve it by the method of separation of variables, and then decompose the heat source $\delta(r, z, t)$ into the related eigenfunctions. Upon separation of variables in the r-direction, the Bessel equation is formed, and the corresponding solution and eigenvalues are derived accordingly. Similarly, the differential equation in the z-coordinate with non-homogeneous boundary conditions can be transformed into one with homogeneous boundary conditions and thus solved by eigenfunction expansions (see, for details, [3–5]). In our present investigation, α_m is the mth positive zero of the following transcendental Eq. (7) for all the cases:

$$J_1(\alpha_m) = 0 \quad (m = 1, 2, 3, \ldots), \tag{7}$$

where $J_\nu(z)$ denotes the familiar Bessel function of the first kind defined by (see, for details, [6])

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{\nu+2n}}{n!\Gamma(\nu + n + 1)} \quad (\nu \in \mathbb{C};\ z \in \mathbb{C} \setminus (-\infty, 0]).$$

Furthermore, the differential equation in the time space can also be solved by the familiar integrating factor method for first-order differential equations with different C_{mn} as follows:

$$u_{mn}(t) = \frac{C_{mn}}{\beta_n^2 + \alpha_m^2} \left(\beta_n^2 e^{-(\beta_n^2 + \alpha_m^2)t} + \alpha_m^2 \right). \tag{8}$$

Then the solution formed by the product of these chosen functions satisfies the heat conduction partial differential Eq. (1) as well as the initial condition and boundary conditions automatically. The Biot number represents the convection condition between solid and fluid interfaces. The more heat convection on the lateral surface and the more thermal energy are efficiently transferred into the surrounding environment through the interface in the condition for a larger value of the Biot number. A constant heat flux condition is shown when the Biot number is infinitesimal. When the Biot number approaches infinity, a constant temperature condition is presented. In this paper, we present different solutions associated with the various convection conditions.

Case 1. Bi = constant; $\text{Bi}_\ell = \text{constant}$.

The corresponding analytical solution is given, in terms of a Fourier–Bessel type series, by

$$u(r, z, t) = \sum_{m=1}^{\infty} \left(zA_m + (\ell - z)B_m + \sum_{n=1}^{\infty} [u_{mn}(t) + D_{mn}] \cdot \left[\cos(\beta_n z) + \frac{\text{Bi}}{\beta_n} \sin(\beta_n z) \right] \right) J_0(\alpha_m r), \tag{9}$$

where the heat source $\delta(x, y, t)$ is decomposed into simple components as given below:

$$\delta(r, z, t) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} D_{mn} \left[\cos(\beta_n z) + \frac{\text{Bi}}{\beta_n} \sin(\beta_n z) \right] \right) J_0(\alpha_m r). \tag{10}$$

Moreover, the heat transfer rate in the z-direction is given by

$$Q(r, z, t) = -\int^r 2\pi r \, \frac{\partial u(r, z, t)}{\partial z} \, dr. \tag{11}$$