Applied Mathematics and Computation 211 (2009) 488–494

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Weak formulation based Haar wavelet method for solving differential equations

Jüri Majak [a,*], Meelis Pohlak [a], Martin Eerme [a], Toomas Lepikult [b]

[a] Department of Mechanics, Tallinn University of Technology, Ehitajate tee 5, 19086 Tallinn, Estonia
[b] The Estonian Information Technology College, Raja 4C, 12616 Tallinn, Estonia

ARTICLE INFO

Keywords:
Haar wavelet
Weak formulation based solution
Shock wave

ABSTRACT

The Haar wavelet based discretization method for solving differential equations is developed. Nonlinear Burgers equation is considered as a test problem. Both, strong and weak formulations based approaches are discussed. The discretization scheme proposed is based on the weak formulation. An attempt is made to combine the advantages of the FEM and Haar wavelets. The obtained numerical results have been validated against a closed form analytical solution as well as FEM results. Good agreement with the exact solution has been observed.

1. Introduction

Burgers equation contains the simplest form of nonlinear advection and dissipation terms and is used for modelling of the wave motion, turbulence, incompressible momentum equations, etc. It is one of the few nonlinear partial differential equations, which has a closed form solution (actually for certain restricted values of kinematic viscosity only). Also, the solution of the Burgers equations has shock wave behaviour when parameter ε vanishes. For that reason, the nonlinear Burgers equation is used most commonly for testing new numerical methods. Large number of papers devoted to numerical solution of the Burgers equation can be found in the literature. The finite difference, finite element and spectral methods are the most widely used numerical techniques for solving nonlinear Burgers equation [1–9]. Treatment of the numerical methods constructed on the method of discretization in time is one of the recent trends [10–13]. A number of various wavelet approaches for solving nonlinear Burgers equation can be found in [14–17].

In [14] and [18] the Haar wavelet method for solving evolution equations is treated. The solution utilized in [14] is based on the integration technique proposed by Lepik in [19]. In latter study the direct method is employed and instead of the solution of the differential equation, its higher order derivative can be expanded into Haar wavelets. Such an approach has been suggested initially by Chen and Hsiao in [20]. A similar approach has been applied by the authors of the current study for solving the limit analysis problems of isotropic plates and shells [21,22]. The Haar wavelet solutions developed by different authors are found to be competitive (simplicity, high accuracy). However, the scope of the problems studied by employing the Haar wavelet based technique is still limited. Engineering problems are not well covered. Most of the Haar wavelet algorithms for solving differential equations are based on strong formulations which is not a common approach for solving engineering problems.

In the current study, the weak formulation based Haar wavelet discretization method (HWDM) is developed. The discretization method proposed is based on the effective properties of the Haar product and coefficient matrices introduced by Hsiao [23] and Chen and Hsiao [20].

* Corresponding author.
E-mail address: jmajak@staff.ttu.ee (J. Majak).

0096-3003/$ - see front matter © 2009 Elsevier Inc. All rights reserved.
doi:10.1016/j.amc.2009.01.089

The Haar wavelet expansion is used for approximation of the unknown function in the initial differential equation as well as for approximation of the weight function. The aim of the current study is to develop the discretization method providing a symmetric discrete system of algebraic equations and preserving the advantages common to Haar wavelets. In the case of FEM, the weight function is determined commonly in accordance with the Galjorkin method. In the case of the proposed Haar wavelet discretization method the same considerations are used, but the Galjorkin method is not applied directly.

In the current paper, the higher and lower order Haar wavelet based approximations are introduced. Decomposition of the Haar wavelet expansion is performed. As a result, the solution is divided into local and global terms.

Computational complexity of the discretization methods appears typically with an increasing number of integration points (mesh). In the case of the Haar wavelet based discretization method we deal with the following problems: calculating the inverse matrix P^{-1} for large order n, limitations for selection of n, uniform grid in standard method application. These limitations can be reduced by dividing the initial integration domain into subdomains. Subdomain based approaches of Haar wavelet based discretization method have been proposed by Cattani [24] and Lepik [19].

In the case of the weak formulation based Haar wavelet discretization method treated in the current study, the computational complexity is reduced due to symmetry of the discrete algebraic system, which allows to apply the Cholesky decomposition.

2. Haar wavelet family

The set of Haar functions is defined as a group of square waves with magnitude ±1 in some intervals and zero elsewhere

$$h_i(t) = \begin{cases} 1 & \text{for } t \in \left[\frac{k}{m}, \frac{k+0.5}{m}\right), \\ -1 & \text{for } t \in \left[\frac{k+0.5}{m}, \frac{k+1}{m}\right), \\ 0 & \text{elsewhere}, \end{cases} \tag{2.1}$$

where $m = 2^j, j \in \{0, 1, \ldots, J\}$, $k = 0, 1 \ldots, m-1$. The integer J determines the maximal level of resolution and the index i is calculated from the formula $i = m + k + 1$.

The Haar functions are orthogonal to each other and form a good transform basis

$$\int_0^1 h_i(t)h_l(t)dt = \begin{cases} 2^{-j}, & i = l = 2^j + k, \\ 0, & i \neq l. \end{cases} \tag{2.2}$$

The Haar matrix is defined through Haar functions as

$$H(t) = \begin{bmatrix} h_1(t) \\ h_2(t) \\ \ldots \\ h_m(t) \end{bmatrix}. \tag{2.3}$$

Any function $y(t)$ that is square integrable and finite in the interval [0, 1) can be expanded into Haar wavelets. It follows from (2.1) that the integration of Haar wavelets results in triangular functions (functions with triangular shape tri(t)=max(1-|t|, 0)). These functions can be expanded into Haar series as

$$\int_0^1 H(t)dt \cong PH(t), \tag{2.4}$$

where P is an $m \times m$ matrix.

The symbolic-numerical algorithm for determining the matrix P is based on the following recursive relation

$$P_m(t) = \frac{1}{2m} \begin{bmatrix} 2mP_{m/2} & -H_{m/2} \\ H_{m/2}^{-1} & 0 \end{bmatrix}. \tag{2.5}$$

In (2.5) H_m indicates the Haar coefficient matrix with elements

$$H_m^{ij} = h_i(t_j), \quad t_j = (2j-1)(2m), \quad j = 1, \ldots, m. \tag{2.6}$$

It follows from definition of the integral operator P that (see (2.3) and (2.4))

$$PH|_{t=0} = [0, 0, \ldots, 0]^T, \quad PH|_{t=1} = [0, 0, \ldots, 0]^T = Y \tag{2.7}$$

The relations ((2.5)–(2.7)) are well known and a quite commonly used for numerical solution of differential equations.

The higher order operational matrices Q, R and S can be introduced as

$$\int_0^1 PH(t)dt \cong QH(t), \quad \int_0^1 QH(t)dt \cong RH(t), \quad \int_0^1 RH(t)dt \cong SH(t). \tag{2.8}$$

It can be verified that the higher order operational matrices Q, R and S as satisfy the following boundary conditions