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Successively iterative method of nonlinear Neumann boundary value problems

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ABSTRACT

Two successively iterative sequences are constructed for computing solutions of the nonlinear Neumann boundary value problems with time singularity. The sequences start off with some constants. Main tool is the fixed point theorem of increasing operator on the order interval. By considering convergence of the sequences, we prove the existence of nontrivial sign-changing solutions.

1. Introduction

Let $\lambda > 0$ be a fixed positive number. In this paper, we consider the following nonlinear second-order Neumann boundary value problem

$$(P)\begin{cases} -u''(t) + \lambda u(t) = f(t, u(t)), & 0 < t < 1, \\ u'(0) = u'(1) = 0. \end{cases}$$

The Neumann boundary value problem (P) can model a class of important physical phenomena whose gradients equal zero in boundary points. The existence of solutions of (P) has been studied widely, for more details, see [2–5,9,10,12] and the references therein. In this paper, we give an iterative technique to obtain nontrivial solutions of the problem (P). The study is based on the following Theorem 1.1 proved in [1], that is, the fixed point theorem of increasing operator on the order interval.

In order to explore the theorem, we briefly recall some definitions. Let X be the Banach space with norm $\|\cdot\|$. A nonempty closed set $K \subset X$ is called *cone* if K satisfies the following conditions: (i) if $x, y \in K$, then $x + y \in K$; (ii) if $x \in K$, then $\mu x \in K$ for any $\mu \geqslant 0$; (iii) if $0 \neq x \in K$, then $-x \notin K$. The cone K is called *normal*, if there exists $\rho > 0$ such that

$$\|x_1 + x_2\| \geqslant \rho, \quad \forall x_1, x_2 \in K, \|x_1\| = \|x_2\| = 1.$$

Let $x_1, x_2 \in X$. We write $x_1 \ll x_2$, if $x_2 - x_1 \in K$. We call the set $[x_1,x_2] = \{x \in X : x_1 \ll x \ll x_2\}$ *order interval* in X.

The operator $T : [x_1,x_2] \to X$ is called *increasing* if $T\bar{x} \ll T\tilde{x}$ for any $\bar{x}, \tilde{x} \in [x_1, x_2]$ and $\bar{x} \ll \tilde{x}$.

Theorem 1.1. *Let X be a Banach space ordered by a normal cone $K \subset X$. Assume that $T : [x_1,x_2] \to X$ is a completely continuous increasing operator such that $x_1 \ll Tx_1$, $Tx_2 \ll x_2$. Then T has a minimal fixed point x^* and a maximal fixed point x^*such that $x_1 \ll x^* \ll x^* \ll x_2$. Moreover, $x^* = \lim_{n\to\infty} T^n x_1$, $x^* = \lim_{n\to\infty} T^n x_2$, where $\{T^n x_1\}_{n=1}^{\infty}$ is an increasing sequence, $\{T^n x_2\}_{n=1}^{\infty}$ is a decreasing sequence.*

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We will apply Theorem 1.1 to construct two successively iterative sequences for the problem (P). The sequences start off with some constants. By considering convergence of the sequences, we will prove an existence theorem of sign-changing solution. In the discussion, we will allow the nonlinear term $f(t,u)$ to be singular at $t = 0$, $t = 1$.

In this paper, the conditions and the results are local. In other words, the existence of solution and iteration only depend upon the growth properties of nonlinear term $f(t,u)$ on some set $(0,1)\times[a,b]$. The localization idea has been widely applied to study the iterations of some nonlinear boundary value problems, for example, see papers [7,8,11,13,14]. However, these papers devote to the positive solutions. The principal purpose of the present work is to study the sign-changing solution of problem (P). As a special case, we consider the positive solution too. We will illustrate that our method is convenient and effective for the nonlinear Neumann boundary value problem (P) by a numerical example.

2. Main results

Let $G(t,s)$ be the Green function of homogeneous linear problem

$$-u''(t)+\lambda u(t)=0,\quad 0\leqslant t\leqslant 1,\quad u'(0)=u'(1)=0.$$

The exact expression of $G(t,s)$ is

$$G(t,s)=\begin{cases}\dfrac{\cosh\left(\sqrt{\lambda}(1-t)\right)\cosh\left(\sqrt{\lambda}s\right)}{\sqrt{\lambda}\sinh\sqrt{\lambda}}, & 0\leqslant s\leqslant t\leqslant 1,\\[2mm] \dfrac{\cosh\left(\sqrt{\lambda}t\right)\cosh\left(\sqrt{\lambda}(1-s)\right)}{\sqrt{\lambda}\sinh\sqrt{\lambda}}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}$$

where $\sinh t=\frac{1}{2}[e^{t}-e^{-t}],\ \cosh t=\frac{1}{2}[e^{t}+e^{-t}]$.

Obviously, $G(t,s)>0$, $0\leqslant t,s\leqslant 1$. Let

$$m=\min_{0\leqslant t,s\leqslant 1}G(t,s),\quad M=\max_{0\leqslant t,s\leqslant 1}G(t,s).$$

Direct computations give that

$$m=G(1,0)=\frac{1}{\sqrt{\lambda}\sinh\sqrt{\lambda}},\quad M=G\left(\frac{1}{2},\frac{1}{2}\right)=\frac{\cosh\sqrt{\lambda}+1}{2\sqrt{\lambda}\sinh\sqrt{\lambda}}.$$

Consider the Banach space $C[0,1]$ with norm $\|u\|=\max_{0\leqslant t\leqslant 1}|u(t)|$. Let

$$K=\{u\in C[0,1]:u(t)\geqslant 0,0\leqslant t\leqslant 1\}.$$

Then K is the cone of nonnegative functions in $C[0,1]$. For u, $v\in C[0,1]$, $u\ll v$ if and only if $u(t)\leqslant v(t)$, $0\leqslant t\leqslant 1$. Moreover, the cone K is normal because $\|u+v\|\geqslant\|u\|=1$ for any u, $v\in K$ and $\|u\|=\|v\|=1$.

We obtain the following existence theorem and iterative schemes.

Theorem 2.1. *Assume that there exist two real numbers a, b such that $a<b$ and the following conditions are satisfied:*

(A1) $f:(0,1)\times[a,b]\to(-\infty,+\infty)$ is continuous and there exists a nonnegative function $j\in C(0,1)\cap L^1[0,1]$ such that

$$|f(t,u)|\leqslant j(t),\quad (t,u)\in(0,1)\times[a,b].$$

(A2) $f(t,\bar{u})\leqslant f(t,\hat{u}),0<t<1,a\leqslant\bar{u}\leqslant\hat{u}\leqslant b$.

(A3) The following inequalities hold

$$m\int_0^1\max\{f(t,a),0\}dt+M\int_0^1\min\{f(t,a),0\}dt\geqslant a,$$

$$M\int_0^1\max\{f(t,b),0\}dt+m\int_0^1\min\{f(t,b),0\}dt\leqslant b.$$

(A4) One of the following conditions is satisfied:

$$(1)a>0;\quad(2)b<0;\quad(3)a\leqslant 0\leqslant b,\quad f(t,0)\not\equiv 0.$$

Then problem (P) has two nontrivial solutions u^*, $v^*\in C^1[0,1]\cap C^2(0,1)$ such that $a\leqslant u^*(t)\leqslant v^*(t)\leqslant b$, $0\leqslant t\leqslant 1$ and $\|u_n-u^*\|\to 0$, $\|v_n-v^*\|\to 0$, where, for $0\leqslant t\leqslant 1$,

$$u_0(t)\equiv a,\quad u_{n+1}(t)=\int_0^1G(t,s)f(s,u_n(s))ds,\quad n=0,1,2,\ldots,$$

$$v_0(t)\equiv b,\quad v_{n+1}(t)=\int_0^1G(t,s)f(s,v_n(s))ds,\quad n=0,1,2,\ldots$$