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A penta-parametric family of fifteenth-order multipoint methods for nonlinear equations

Young Hee Geum [1], Young Ik Kim *

Department of Applied Mathematics, Dankook University, Cheonan, 330-714, Republic of Korea

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ABSTRACT

A penta-parametric family of four-step multipoint iterative methods of order fifteen for nonlinear algebraic equations are developed and their convergence properties are established. The efficiency indices are all found to be $15^{1/5} \approx 1.71877$, better than $14^{1/5} \approx 1.69522$ of a family of fourteenth-order methods suggested by Neta [9]. Numerical examples are demonstrated to verify the developed theory.

1. Introduction

Recently eighth-order 3-step multipoint iterative methods to find a numerical solution of a nonlinear algebraic equation $f(x) = 0$ have been developed by Bi-Ren-Wu [2], Bi-Wu-Ren [3], Geum-Kim [4], Liu-Wang [8] and Wang-Liu [11]. These methods have efficiency index [10] of $8^{1/4}$, being free from second derivatives. The 2nd-step of these methods frequently uses King's fourth-order method [6] and Jarratt's fourth-order method [5]. Neta [9] suggested a family of fourteenth-order multipoint iterative methods with an efficiency index of $14^{1/5}$ which are introduced here:

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - \dfrac{f(x_n) + Af(y_n)}{f(x_n) + (A-2)f(y_n)} \dfrac{f(y_n)}{f'(x_n)}, A \in \mathbb{R}, \\ s_n = z_n - \dfrac{f(x_n) - f(y_n)}{f(x_n) - 3f(y_n)} \cdot \dfrac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} + \theta_1 f^2(x_n) + \theta_2 f^3(x_n) + \theta_3 f^4(x_n), \end{cases} \tag{1.1}$$

where $\theta_3 = \frac{\Delta_1 - \Delta_2}{F_s - F_y}, \theta_2 = -\Delta_1 + \theta_3(F_s + F_z), \quad \theta_1 = \phi_s + \theta_2 F_s - \theta_3 F_s^2$ with $\Delta_1 = \frac{\phi_s - \phi_z}{F_s - F_z}, \Delta_2 = \frac{\phi_y - \phi_z}{F_y - F_z}, \phi_s = \frac{1}{F_s}\left(\frac{s_n - x_n}{F_s} - \frac{1}{f'(x_n)}\right), \phi_y = \frac{1}{F_y}\left(\frac{y_n - x_n}{F_y} - \frac{1}{f'(x_n)}\right), \phi_z = \frac{1}{F_z}\left(\frac{z_n - x_n}{F_z} - \frac{1}{f'(x_n)}\right), F_s = f(s_n) - f(x_n), F_y = f(y_n) - f(x_n)$ and $F_z = f(z_n) - f(x_n)$.

Notice that the fourth equation of (1.1) is obtained by means of inverse interpolation [10] and the coefficient $\theta_i (i = 1,2,3)$ is dependent upon the values of x_n, y_n, z_n, s_n, $f(x_n)$, $f(y_n)$, $f(z_n)$, $f(s_n)$ as well as a single derivative $f'(x_n)$. This kind of function

* Corresponding author.
E-mail addresses: conpana@empal.com (Y.H. Geum), yikbell@yahoo.co.kr (Y.I. Kim).
[1] Instructor of Mathematics.

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dependence often prohibits us from reducing computational time to implement iterative scheme (1.1). Although Neta did not provide an explicit form of the error equation of (1.1), we successfully find the corresponding error equation below:

$$e_{n+1} = -c_2^3 c_3 \{-(1+2A)c_2^2 + c_3\}^2 (14c_2^4 - 21c_2^2 c_3 + 3c_3^2 + 6c_2 c_4 - c_5) e_n^{14} + O(e_n^{15}). \tag{1.2}$$

The main aim is to develop a higher-order root-finding method that deals with complex-valued as well as real-valued nonlinear algebraic equations. We now assume that $f : \mathbb{C} \to \mathbb{C}$ has a simple root α and is analytic [1] in a region containing α. To avoid the sophisticated function-dependent coefficients $\theta_i (i = 1,2,3)$ in (1.1), we introduce constant control parameters to propose a new family of four-step multipoint methods of order fifteen stated as follows: for $n = 0,1,\ldots,$

$$\begin{cases} y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}, \\ z_n = y_n - K_f(u_n)\dfrac{f(y_n)}{f'(x_n)}, \\ s_n = z_n - H_f(u_n, v_n, w_n)\dfrac{f(z_n)}{f'(x_n)}, \\ x_{n+1} = s_n - W_f(u_n, v_n, w_n, t_n)\dfrac{f(s_n)}{f'(x_n)}, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} K_f(u_n) &= \frac{1 + \beta u_n + \frac{\beta}{2} u_n^2}{1 + (\beta - 2)u_n - (1 + \frac{5\beta}{2})u_n^2}, \\ H_f(u_n, v_n, w_n) &= \frac{1 - u_n - \frac{3}{2} v_n + (\sigma - 4)w_n}{1 - 3u_n - \frac{5}{2} v_n + \sigma w_n}, \\ W_f(u_n, v_n, w_n, t_n) &= \frac{1 - u_n - \frac{3}{2} v_n + (\sigma - \frac{9}{2})w_n + \frac{3}{2} t_n + (\sigma - \frac{19}{4}) v_n w_n + \Gamma v_n^3 + \Psi v_n u_n^4 + \Omega t_n u_n^2}{1 - 3u_n - \frac{5}{2} v_n - (\sigma - \frac{1}{2})w_n + \frac{1}{2} t_n + (\sigma - \frac{25}{4}) v_n w_n + (\Gamma - \frac{3}{2}) v_n^3 + (\Psi + \frac{\beta}{2}) v_n u_n^4 + (\Omega - 3) t_n u_n^2}, \end{aligned} \tag{1.4}$$

with five constant real parameters β, σ, Γ, Ψ, Ω to be chosen freely, and

$$u_n = f(y_n)/f(x_n), \quad v_n = f(z_n)/f(y_n), \quad w_n = f(z_n)/f(x_n), \quad t_n = f(s_n)/f(z_n). \tag{1.5}$$

It is not difficult to show that

$$y_n = \alpha + O(e_n^2), \quad z_n = \alpha + O(e_n^4), \quad s_n = \alpha + O(e_n^8), \tag{1.6}$$

$$u_n = O(e_n), \quad v_n = O(e_n^2), \quad w_n = O(e_n^3), \quad t_n = O(e_n^4), \tag{1.7}$$

$$\frac{f(x_n)}{f'(x_n)} = O(e_n), \quad \frac{f(y_n)}{f'(x_n)} = O(e_n^2), \quad \frac{f(z_n)}{f'(x_n)} = O(e_n^4), \quad \frac{f(s_n)}{f'(x_n)} = O(e_n^8). \tag{1.8}$$

We now introduce more general parameters λ, μ, γ, a, b, c, d in two expressions of (1.4) to get:

$$\begin{aligned} K_f(u_n) &= \frac{1 + \beta u_n + \lambda u_n^2}{1 + (\beta - 2)u_n + \mu u_n^2}, \\ H_f(u_n, v_n, w_n) &= \frac{1 + a u_n + b v_n + \gamma w_n}{1 + c u_n + d v_n + \sigma w_n}, \end{aligned} \tag{1.9}$$

According to (1.7), W_f in (1.4) can be written in the following form:

$$W_f(u_n, v_n, w_n, t_n) = 1 + p_1 e_n + p_2 e_n^2 + p_3 e_n^3 + p_4 e_n^4 + p_5 e_n^5 + p_6 e_n^6 + O(e_n^7), \tag{1.10}$$

where $p_j = p_j(\lambda, \mu, \gamma, a, b, c, d, \beta, \sigma, \Gamma, \Psi, \Omega)$ depends on constant control parameters λ, μ, γ, a, b, c, d, β, σ, Γ, Ψ, Ω for each j ranging from 1 to 6. We further introduce more general parameters $K_1, K_2, \ldots, K_8$, $\rho, \Gamma, \Psi, \Omega, \hat{\rho}, \widehat{\Gamma}, \widehat{\Psi}, \widehat{\Omega}$ in W_f described by (1.10) to take the following form:

$$W_f(u_n, v_n, w_n, t_n) = \frac{1 + K_1 u_n + K_2 v_n + K_3 w_n + K_4 t_n + \rho v_n w_n + \Gamma v_n^3 + \Psi v_n u_n^4 + \Omega t_n u_n^2}{1 + K_5 u_n + K_6 v_n + K_7 w_n + K_8 t_n + \hat{\rho} v_n w_n + \widehat{\Gamma} v_n^3 + \widehat{\Psi} v_n u_n^4 + \widehat{\Omega} t_n u_n^2}. \tag{1.11}$$

Then through an analysis to be shown in Section 2, the desired form of W_f in (1.4) will be obtained along with the derivation of the corresponding error equation stating convergence order of fifteen.

Observe that (1.3) requires five new function evaluations for $f(x_n)$, $f(y_n)$, $f(z_n)$, $f(s_n)$ and $f'(x_n)$ per iteration. The main objective of this paper is to find relationships among these parameters so that iterative scheme (1.3) with (1.4) has fifteenth-order convergence with five constant control parameters to be freely chosen, and its efficiency index [9] is $15^{1/5} \approx 1.71877$. In addition, deriving the asymptotic error constant or error equation is another goal of this paper. To measure convergence behavior within a given error bound, the values of $|x_n - \alpha|$ as well as CPU times of proposed methods (1.3) with (1.4) will be compared with those of iterative methods (1.1). Typical forms of $W_f(u_n, v_n, w_n, t_n)$ are displayed in Section 2. Numerical examples are presented in Section 3 to verify the underlying theory developed in this paper.