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Hopf bifurcations in a Ricardo-Malthus model

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ABSTRACT

Brander and Taylor presented a simple and basic framework for discussing the problem on human population and renewable natural resources in the year 1998, and D'Alessandro recently extended this work mainly by introducing a nonlinear term into the model, if seeing from the mathematical point of view. A limit cycle in this new model was reported by the author via numerically simulated drawing. In this paper, we show that this limit cycle actually is a bifurcating limit cycle of a one-parameter Hopf bifurcation.

1. Introduction

Brander and Taylor [1] presented a simple and basic framework for discussing the problem on human population and renewable natural resources, relative to the Lotka-Volterra predator–prey model with man as the predator and the resources as the prey, in the year 1998. The model follows Malthusian population dynamics [2] and Ricardian production structure [3], and so is called a Ricardo-Malthus model. The original model is a dynamical system as follows:

$$\begin{cases} \dot{S} = [r(1 - S/K) - \alpha\beta L]S, \\ \dot{L} = (b - d + \phi\alpha\beta S)L, \end{cases}$$

where S is the resource stock, L is the labor force (or the population), r is the intrinsic growth rate (or the regeneration rate), K is the carrying capacity of resource, α is the technological parameter of resource harvesting, β is the share of individual consumption on the resource good, b and d are the birth and death rates of population respectively, and ϕ is the fertility parameter of population. This model shows that the over-exploitation of natural resources causes a sharp reduction in the human population. Several other authors have developed this model by taking into account additional aspects such as institutions, property rights, and technical progress (see [4–13]).

D'Alessandro [14] recently extended Brander and Taylor's work to account for the heterogeneity of historical human development paths. In this extension, there are a renewable resource, forest, and an inexhaustible resource, land, and the production, at same time, is extended to be having wood and corn. The extended model is as follows:

$$\begin{cases} \dot{S} = [\rho(S/\underline{K} - 1)(1 - S/\overline{K}) - \alpha\beta L]S, \\ \dot{L} = \gamma[\lambda(1 - \beta)^{\delta} L^{\delta - 1} + \bar{\phi}\alpha\beta S - \bar{\sigma}]L, \end{cases}$$

where S is the stock of forest, L is the labor force (or the population), $\underline{K}$ and $\overline{K}$ are the lower threshold quantity and the carrying capacity of forest, α is the technological parameter of resource harvesting, β is the parameter of preferences on consumption, γ is the caloric unit of corn, λ is the index of land fertility, δ is the technological parameter of land, $\overline{\phi}$ and $\overline{\sigma}$ are the

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wood and per capita natural caloric levels expressed in terms of mass units of corn respectively, and ρ is the intrinsic regeneration rate of forest.

The new model has more complex dynamics than Brander and Taylor's. Accordingly, the author introduces a reasonable nonlinear term $\lambda(1-\beta)^{\delta}L^{\delta-1}$ as well as increasing difficulty in mathematics into the model. In contrast to Brander and Taylor's model which has no Hopf bifurcation as well as no limit cycle, a limit cycle found via numerically simulated drawing was reported in this new model, but without strict mathematical proof. As a part of the work of the Ricardo-Malthus model, we focus on the Hopf bifurcation and do not intend to study other things in this paper. Using Hopf bifurcation theorem and the normal form theory, we are to show that this reported limit cycle actually is a bifurcating limit cycle of a one-parameter Hopf bifurcation of the model.

The rest of this paper is organized as follows. Section 2 presents the Hopf bifurcation theorem. Section 3 shows the existence of Hopf bifurcation of the model. Section 4 gives numerical examples. And Section 5 concludes.

2. The Hopf bifurcation theorem

We first introduce the Hopf bifurcation theorem.

Theorem 1 (Hopf [15]). *Suppose that the system $\dot{\boldsymbol{x}} = f_{\mu}(\boldsymbol{x}), \boldsymbol{x} = (x, y, \ldots)^T \in \mathbb{R}^n,\ \mu \in \mathbb{R}$ has an equilibrium (x_0, μ_0) at which the following properties are satisfied:*

(H1) *$D_{\boldsymbol{x}} f_{\mu_0}(\boldsymbol{x}_0)$ has a simple pair of pure imaginary eigenvalues and no other eigenvalues with zero real parts. Then* (H1) *implies that there is a smooth curve of equilibria $(x(\mu), \mu))$* with $x(\mu_0) = x_0$. *The eigenvalues $\lambda(\mu),\ \bar{\lambda}(\mu)$ of $D_{\boldsymbol{x}} f_{\mu_0}(\boldsymbol{x}(\mu))$* which are *imaginary at $\mu = \mu_0$ vary smoothly with μ. If, moreover,*

(H2)

$$d = \frac{\mathrm{d}}{\mathrm{d}\mu} \mathrm{Re}\lambda(\mu)\Big|_{\mu=\mu_0} \neq 0,$$

then there is a unique three dimensional center manifold passing through (x_0, μ_0) in $\mathbb{R}^n \times \mathbb{R}$ and a smooth system of coordinates (*preserving* the planes $\mu = const.$) *for which the Taylor expansion of degree* 3 *on the center manifold is given by*

$$\dot{x} = (d\mu + a(x^2 + y^2))x + (\omega + c\mu + b(x^2 + y^2))y,$$
$$\dot{y} = (\omega + c\mu + b(x^2 + y^2)x + (d\mu + a(x^2 + y^2))y.$$

If $a \neq 0$, there is a surface of periodic solutions in the center manifold which has quadratic tangency with eigenspace of $\lambda(\mu_0),\ \bar{\lambda}(\mu_0)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, *then these periodic solutions are stable limit cycles, while if $a > 0$, the periodic solutions are repelling.*

3. Existence of Hopf bifurcations in D'Alessandro's model

For convenience, we change some of the notations in the original model and use our own ones. The model is then rewritten as follows:

$$\begin{cases} \dot{x} = [\rho(x/k_1 - 1)(1 - x/k_2) - \alpha\beta y]x, \\ \dot{y} = \gamma[\lambda(1-\beta)^{\delta} y^{\delta-1} + k_3\alpha\beta x - k_4]y, \end{cases} \tag{1}$$

where $\alpha > 0$, $\beta \in (0,1)$, $\gamma > 0$, $\lambda > 0$, $\delta \in (0,1)$, $\rho > 0$, $k_1 > 0$, $k_2 > 0$, $k_1 < k_2$, $k_3 > 0$, and $k_4 > 0$. Letting $\dot{x} = 0$ and $\dot{y} = 0$ and ignoring the equilibria on the axes, we have

$$\rho(x/k_1 - 1)(1 - x/k_2) - \alpha\beta y = 0, \tag{2}$$
$$\lambda(1-\beta)^{\delta} y^{\delta-1} + k_3\alpha\beta x - k_4 = 0. \tag{3}$$

The internal equilibria in the first quadrant are the solutions of the system (2), (3), denoted by $Q^*(x^*, y^*)$. The Jacobian of system (1) evaluated at Q^* is

$$\boldsymbol{J} = \begin{pmatrix} -\frac{3\rho}{k_1k_2}x^{*2} + 2\frac{(k_1+k_2)\rho}{k_1k_2}x^* - \rho - \alpha\beta y^* & -\alpha\beta x^* \\ \gamma k_3\alpha\beta y^* & \gamma[\lambda\delta(1-\beta)^{\delta} y^{*(\delta-1)} + k_3\alpha\beta x^* - k_4] \end{pmatrix}. \tag{4}$$

Thus, the trace of $\boldsymbol{J}$ is

$$\mathrm{TR} = \mathrm{trace}\,\boldsymbol{J} = -\frac{3\rho}{k_1k_2}x^{*2} + 2\frac{(k_1+k_2)\rho}{k_1k_2}x^* - \rho - \alpha\beta y^* + \gamma[\lambda\delta(1-\beta)^{\delta} y^{*(\delta-1)} + k_3\alpha\beta x^* - k_4], \tag{5}$$