



Invariant norm quantifying nonlinear content of Hamiltonian systems

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ABSTRACT

Given a Hamiltonian system, one can represent it using a symplectic map. This symplectic map is specified by a set of homogeneous polynomials which are uniquely determined by the Hamiltonian. In this paper, we construct an invariant norm in the space of homogeneous polynomials of a given degree. This norm is a function of parameters characterizing the original Hamiltonian system. Such a norm has several potential applications.

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1. Introduction

Hamiltonian systems form an important class of dynamical systems. It is important to be able to quantify the nonlinear content of Hamiltonian systems through an appropriate quantity. This can then serve as a merit function for optimizing the performance of the Hamiltonian system as a function of the parameters characterizing the Hamiltonian system. In particular, one can try to maximize the stability region of the system (in the context of particle accelerators [6]), minimize optical aberrations (in the context of optical systems [8]), etc. We quantify the nonlinear content of the Hamiltonian system using its corresponding symplectic map.

Symplectic maps and invariants have been successfully used to better understand the Hamiltonian systems [1–3,5,6,9–11,13–20,26,27]. In fact, the time evolution of the Hamiltonian system with $2n$ degrees of freedom can be directly described by the symplectic map, say $\mathcal{M}$, generally specified by a set of homogeneous polynomials in the $2n$ phase space variables of the Hamiltonian system and of a certain degree [9]. The nonlinear content of $\mathcal{M}$ (and hence of the original Hamiltonian system) is given by homogeneous polynomials of degree greater than two. However, this representation is not very useful to explicitly compute the nonlinear content as it involves an infinite series. This limitation is overcome by employing Lie perturbation theory where we consider the terms degree by degree in the space of homogeneous polynomials. Further, we require that the quantity characterizing the nonlinear content be invariant under the action of an appropriate symmetry group. Here, the underlying symmetry group for the full $\mathcal{M}$ turns out to be an infinite dimensional, non-compact, Lie group. This implies that it is not possible to construct a quantity invariant under the full symmetry group. Consequently, rather than quantifying and minimizing the entire nonlinear content in $\mathcal{M}$, we will in fact restrict ourselves to doing the same for symplectic maps with homogeneous polynomials truncated at degree 3. This is in line with the perturbative approach that we have adopted since the leading order nonlinearity in $\mathcal{M}$ comes from homogeneous polynomials of degree 3. Furthermore, as we shall see later, for a quantity to remain invariant under the action of the above truncated symplectic map, it is sufficient to demand that it remains invariant under the linear part of the symplectic map since the nonlinear term contributes only a fourth order correction that we can ignore (in the spirit of perturbation theory) having truncated at degree 3. By this process, we have also reduced the dimension of the underlying symmetry group to be the finite dimensional real

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symplectic group $Sp(2n, \mathbb{R})$. Since even this group is non-compact, we still are not in a position to define any quantity that is invariant under its action. To get around this problem, we put the truncated symplectic map into what is known as its “normal form” using a symplectic (canonical) transformation. Now, the symmetry group for the normal form is the group of real symplectic orthogonal matrices. We note that this is isomorphic to the compact unitary group $U(n)$. Hence, we can now restrict ourselves to finding a quantity in the space of homogenous polynomials of degree 3 in the $2n$ phase space variables that is invariant under the action of the above unitary group. This is accomplished by performing an invariant integration over $U(n)$ of a suitable function. An earlier, approximate treatment can be found in [28]. Once we have an invariant, we can use this as a merit function to determine optimal values of the system parameters and consequently, enhance the “performance” of the system.

2. Symplectic maps and homogeneous polynomials

Through our discussion in the first section, it is quite clear that the homogenous polynomials of the phase space variables play a vital role in the Lie perturbation theory of Hamiltonian dynamics. In this section, we shall fix notations and build the theory to that extent in order that we give a meaningful representation of any homogenous polynomial of a certain degree in terms of the basis monomials of the appropriate space.

Consider the Hamiltonian system given by $2n$ phase space variables that we denote by

$$z = (q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n).$$

For any fixed phase space function $f(z)$, let $\mathcal{L}_f$ denote the corresponding Lie operator defined on the space of phase space functions by

$$\mathcal{L}_f g := [f, g] = fg - gf.$$

It is then a simple observation that the above defined bracket is anti-symmetric and linear that satisfies $[f, (gh)] = [f, g]h + g[f, h]$. The exponential of the above defined operator is called the Lie transformation that is again defined on the space of phase space functions.

$$(e^{\mathcal{L}_f})(g) := \left(\sum_{n \geq 0} \frac{\mathcal{L}_f^n}{n!} \right)(g) = \frac{g}{0!} + \frac{[f, g]}{1!} + \frac{[f, [f, g]]}{2!} + \frac{[f, [f, [f, g]]]}{3!} + \dots$$

The effect of the Hamiltonian system on a particle can be formally expressed as the action of a map $\mathcal{M}$ that takes the particle from its initial state z_{in} to its final state z_{fin} , i.e., $z_{\text{fin}} = \mathcal{M}z_{\text{in}}$. It can be shown that $\mathcal{M}$ is a symplectic map, [6,9]. Symplectic maps are maps whose $2n \times 2n$ Jacobian matrices $M(z)$ satisfy the ‘symplectic condition’ given by $\tilde{M}(z)JM(z) = J$, where $\tilde{M}$ represents the transpose of M and J is the fundamental symplectic matrix given by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Here I is an $n \times n$ identity matrix. The set of all symplectic matrices form the real symplectic group $Sp(2n, \mathbb{R})$. This finite dimensional non-compact real symplectic Lie group is the underlying symmetry group for the linear part of the symplectic map $\mathcal{M}$ with n degrees of freedom. The symplectic map can be factorised using the result due to Dragt and Finn.

Lemma 1 (Dragt–Finn Factorisation Theorem, [7]). *The symplectic map $\mathcal{M}$ can be factorised as*

$$\mathcal{M} = \hat{M} e^{\mathcal{L}_{f_3}} e^{\mathcal{L}_{f_4}} \dots e^{\mathcal{L}_{f_n}} \dots,$$

where, $\hat{M}$ gives the linear part of the map while the infinite product of Lie transformations $e^{\mathcal{L}_{f_n}}$, $n = 3, 4, \dots$ represents the nonlinear part of $\mathcal{M}$. Here, f_n denotes a unique homogenous polynomial in the phase space variables of degree n .

Observe that the linear part $\hat{M}$ of the symplectic map $\mathcal{M}$ has an equivalent representation in terms of the Jacobian matrix M of the map $\mathcal{M}$,

$$\hat{M}z_i = M_{ij}z_j = (Mz)_i.$$

We now undertake the task of indexing the basis monomials appropriately. Although this is a simple task, we urge the reader to look at its significance in the later sections. Let us denote by $\mathcal{P}^{(m)}$, the space of all homogenous polynomials in z of degree m . Let $\{P_\alpha^{(m)}\}$ be the basis for this space. By a result due to Nijenhuis and Wilf [24], we know that the dimension of this space is given by

$$N(2n, m) = \binom{2n + m - 1}{m}.$$

We take the basis $P_\alpha^{(m)}(z)$ to be the basis monomials of degree m in the $2n$ variables,

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