Applied Mathematics and Computation 217 (2010) 2737–2749

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical performance of parallel group explicit solvers for the solution of fourth order elliptic equations

Norhashidah Hj. Mohd. Ali *, Khoo Kok Teong

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Penang, Malaysia

ARTICLE INFO

Keywords:
Parallel implementation
Biharmonic equation
Group explicit methods
Message passing interface

ABSTRACT

Many applications in applied mathematics and engineering involve numerical solutions of partial differential equations (PDEs). Various discretisation procedures such as the finite difference method result in a problem of solving large, sparse systems of linear equations. In this paper, a group iterative numerical scheme based on the rotated (skewed) five-point finite difference discretisation is proposed for the solution of a fourth order elliptic PDE which represents physical situations in fluid mechanics and elasticity. The rotated approximation formulas lead to schemes with lower computational complexities compared to the centred approximation formulas since the iterative procedure need only involve nodes on half of the total grid points in the solution domain. We describe the development of the parallel group iterative scheme on a cluster of distributed memory parallel computer using Message-Passing Interface (MPI) programming environment. A comparative study with another group iterative scheme derived from the centred difference formula is also presented. A detailed performance analysis of the parallel implementations of both group methods will be reported and discussed.

1. Introduction

Consider the biharmonic equation which is an elliptic, fourth order PDE of the form:

$$\nabla^4 v = \frac{\partial^4 v}{\partial x^4} + 2\frac{\partial^4 v}{\partial x^2 \partial y^2} + \frac{\partial^4 v}{\partial y^4} = c, \quad (x,y) \in \Omega, \tag{1.1}$$

where c is a continuous function of two variables and $\Omega = \{(x,y) : 0 \leqslant x, y \leqslant L\}$. We assume v and $\frac{\partial v}{\partial n}$, which is the derivative in the direction of the outward normal to the boundary $\partial\Omega$, are defined at the boundary $\partial\Omega$. We also assume Eq. (1.1) to be subjected to the boundary condition $v = 0$ and $\frac{\partial^2 v}{\partial n^2} = 0$ along its four edges. This case corresponds to a plate with simply supported edges [16]. By introducing the variable $\nabla^2 v = u$ [15], Eq. (1.1) can be reduced to a coupled Poisson equations

$$\nabla^2 u = c, \quad \text{with } u = 0 \text{ along the boundary } \partial\Omega, \tag{1.2}$$

$$\nabla^2 v = u, \quad \text{with } v = 0 \text{ along the boundary } \partial\Omega. \tag{1.3}$$

Note that both Eqs. (1.2) and (1.3) are of the form

$$\nabla^2 \phi = \psi(x,y), \quad (x,y) \in \Omega, \tag{1.4}$$

* Corresponding author.
E-mail address: shidah@cs.usm.my (N.Hj. Mohd. Ali).

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doi:10.1016/j.amc.2010.08.009

which is the usual Poisson equation. If we solve (1.2) for u and then use this known value to solve (1.3) for v, then the solution of this equation will represent the solution of Eq. (1.1) with the prescribed boundary conditions. There is a number of other equally relevant cases of boundary conditions which would lead to a different number of unknowns in Eqs. (1.2) and (1.3). In such cases, a methodology introduced in Mihajlovic and Silvester [12] will provide a framework for recovering the square discrete Poisson operators. Thus, the algorithm for the solution of (1.2)–(1.3) represents a special case of this general methodology for the solution of the mixed biharmonic problem.

With the advancement in parallel computer systems, the concept of group explicit methods have been widely explored since the early 1980s due to their amenability to be used on this type of architectures [1–3,7–9,11,13,14,18,19]. In their early work, Evans and Abdullah [9] generated single-step, one-parameter families of finite difference approximations to the heat equation in one space dimension by coupling in groups of two the values of the approximations obtained by known asymmetric formulas at adjacent grid points at the advanced time level. The resulting equations are implicit but they can be easily converted to explicit form. The method was shown to possess unconditional stability with good accuracies. Evans and Abdullah [8] also developed the group explicit method for the solution the two dimensional diffusion where a general two-level six point finite difference approximation was developed to solve the parabolic equation. In solving the Poisson equation, Yousif and Evans [18] developed the Explicit Group (EG) scheme which uses a small fixed size group strategy accelerated by the Successive OverRelaxation (S.O.R.) iterative method and was shown to require fewer arithmetic operations and are simpler to program than the classic 1-line block iterative method [7,18]. The construction of this new grouping of the mesh points into smaller size groups of 2, 4, 6, 9, 12, 16, and 25 points were extensively investigated where analysis indicated that the gains in execution timings of the four-point EG method over the 1-line smoother ranges from 25–36% [18]. Since then, the emergence of newer explicit group methods with promising and improved results was greatly observed. Among them are the works of Abdullah [1] who developed the four-point Explicit Decoupled Group (EDG) by discretising the PDEs on rotated grids. Yousif and Evans [19] extended the method to six and nine-point EDG method in solving the second order Poisson equation. These new iterative algorithms have been developed to run on shared memory parallel computers where near linear speedups were observed [19]. In a recent work, Ali and Lee [3] derived the Accelerated OverRelaxation (AOR) variant of the EDG group scheme in the solution of elliptic equation where its performance results were compared with the EG (AOR) proposed by Martins et al. [11].

To date there has not been any detailed study on the performance models of the EG and EDG schemes in solving a coupled elliptic equations on a message passing environment. The suitability of these group relaxation algorithms on the PDEs of the forms (1.2)–(1.3) on this type of platform is still an open question. The main objective of this paper is to perform a detailed performance analysis of the parallel implementation for the EG and EDG methods in solving the coupled Eqs. (1.2)–(1.3) in an MPI environment. A brief description of the group finite difference approximations in solving the biharmonic equation is presented in Section 2 followed by the operation count analysis in Section 3. Section 4 describes the parallelising strategies used on these methods while Section 5 presents the theoretical estimates of the complexity analysis of the sequential and parallel implementations. Numerical experiments carried out and the discussion of results are summarized in Sections 6 and 7 respectively. The concluding remarks is presented in Section 8.

2. The group iterative methods

Let Ω be discretised uniformly in both x and y directions with the mesh size $h = \Delta x = \Delta y = L/n$, where n is an integer. The most basic finite difference approximation in solving the Poisson problem is by using the centred five-point formula. Eq. (1.4) can then be discretised about the point (x_i, y_i) as follows:

$$\phi_{i,j+1} + \phi_{i,j-1} + \phi_{i+1,j} + \phi_{i-1,j} - 4\phi_{i,j} = h^2\psi_{i,j}, \tag{2.1}$$

where $x_i = ih$, $y_j = jh$ $(i,j = 0,1,2,\ldots,n)$ and $\phi(x_i, y_j) = \phi_{ij}$. It should be noted however that this type of discretisation may not be the best choice for biharmonic problem on the domains with sufficiently small corners since they exhibit small-scale phenomena as discussed in Bjostrad and Tjostheim [4] and Brown et al. [5]. In such situation, adaptively refined or unstructured grids and the finite element discretisation will be more appropriate.

An alternative approximation to Eq. (1.4) may be derived from the rotated (skewed) five-point finite difference formula [6] to give

$$\phi_{i+1,j+1} + \phi_{i-1,j-1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} - 4\phi_{i,j} = 2h^2\psi_{i,j}. \tag{2.2}$$

It can be observed that Eq. (2.2) may be obtained from Eq. (2.1) by rotating the computational molecule of the centred five-point operator clockwise by 45°. The group iterative schemes under study are derived from these finite difference formulas and will be described in the following subsections.

2.1. The four-point explicit group (EG) method

Assuming n to be odd, the mesh points are grouped in blocks of four points and Eq. (2.1) applied to each of these points resulting in a (4×4) system of equations of the form [7,18]: