Applied Mathematics and Computation 217 (2010) 2788–2797

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Time-stepping in Petrov–Galerkin methods based on cubic B-splines for compactons

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ARTICLE INFO

Keywords:
Compacton
Rosenau–Hyman equation
Numerical methods
Time integration

ABSTRACT

Four numerical methods with first- to fourth-order of accuracy have been developed for the time integration of the Rosenau–Hyman $K(2,2)$ equation. The error in the solution and the invariants for the propagation of one-compacton, and the stability in collisions among compactons have been studied using these methods. Numerically-induced radiation has also been characterized by means of wavefront velocity and wavefront amplitude, showing that the self-similarity of the radiation wavepackets observed in the numerical results is a consequence of the time-stepping method. Among the four methods studied in this paper, the best results in terms of accuracy, computational cost, and stability have been obtained by means of using the second-order time integration method.

1. Introduction

Compactons are traveling wave solutions with compact support resulting from the balance of both nonlinearity and nonlinear dispersion. These solutions were first found by Rosenau and Hyman [1] in the (focusing) $K(2,2)$ compacton equation given by:

$$u_t - c_0 u_x + (u^2)_x + (u^2)_{xxx} = 0, \quad (1)$$

where $u(x,t)$ is the wave amplitude, x is the spatial coordinate, t is time, c_0 is a constant velocity, and the subindexes indicate differentiation. There is a large number of nonlinear evolution equations with compacton solutions [2]. These solutions can be calculated by several analytical means, like Adomian [3], bifurcation [4], transformation [5], and variational [6] methods. However, for a general nonlinear evolution equation only numerical methods can be applied in order to determine its solutions.

The numerical solution of nonlinear evolution equations with compactons, such Eq. (1), is a very challenging problem, but the reasons behind these numerical difficulties have not been currently explained [7–9]. The most widely used numerical methods in space are pseudospectral ones [1,10]. These methods require artificial dissipation (hyperviscosity) using high-pass filters. Finite element methods based on cubic B-splines [7] and on piecewise polynomials discontinuous at the element interfaces [11], second-order finite difference methods [12,13], high-order Padé methods [14,15], modified equation methods [16], and the method of lines with adaptive mesh refinement [9] have also been used with success. These methods also require artificial dissipation to simulate interacting compactons. Finally, particle methods based on the dispersive-velocity method were also developed [10].

Up to the authors' knowledge, a comparison of time integration methods for equations with compacton solutions has not been developed in the past. However, time-stepping is a very important factor in the numerical accuracy and the stability of

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doi:10.1016/j.amc.2010.08.013

numerical methods for nonlinear evolution equations [17,18]. Moreover, the effect of the time integration method in the self-similarity of the numerically induced radiation observed in Ref. [19] also requires further attention. In this paper, four numerical methods for time integration have been developed and compared when using a Petrov–Galerkin, finite element method in space for the $K(2,2)$ equation. Accuracy, invariant conservation, stability, numerically-induced radiation, and robustness in compacton collisions have been studied as function of both compacton and method parameters.

The contents of this paper are as follows. Next section presents four numerical methods with first, second, third, and fourth-order of accuracy in time, using the same spatial integration. Section 3 summarizes the analysis and practical behavior of the methods considered in this paper; concretely, Section 3.1 study the propagation of one-compacton solutions, Section 3.2 devotes to the numerically-induced radiation, and Section 3.3 addresses the comparison among Methods 1–4 for problems with multiple colliding compactons. Finally, the last section is devoted to some conclusions.

2. Numerical methods

Let us consider the numerical solution of Eq. (1) by means of a Petrov–Galerkin approximation in space with periodic boundary conditions, using C^0 continuous piecewise linear interpolants and C^2 continuous Schoenberg cubic B-splines, respectively, as trial and test functions. For the nonlinear terms, the product approximation is applied [7]. The resulting weak formulation for Eq. (1) is as follows: Find a function

$$U(x,t) = \sum_{i=0}^{N} U_i(t)\phi_i(x),$$

such that

$$\langle U_t, \psi_j \rangle - c_0 \langle U_x, \psi_j \rangle + \langle (U^2)_x, \psi_j \rangle + \langle (U^2)_x, (\psi_j)_{xx} \rangle = 0, \tag{2}$$

for all $\psi_j(x)$, $j = 0,1,\ldots,N$, where a uniform mesh is used, $x_i = x_0 + i\Delta x$, $U_i(t)$ is an approximation to the exact solution $u(x_i,t)$, $\phi_i(x)$ are the usual piecewise linear hat functions associated with the node x_i, i.e., $\phi_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta), $\psi_j(x)$ are cubic B-splines defined in a $4\Delta x$ interval, which are C^2 continuous as required by Eq. (2), and, finally, the inner product is

$$\langle f, g \rangle = \int_{x_0}^{x_N} f(x)g(x)dx.$$

The evaluation of the inner products in Eq. (2) yields the following system of ordinary differential equations

$$\mathcal{A}(\mathrm{E})\frac{dU_i}{dt} - c_0\mathcal{B}(\mathrm{E})U_i + \mathcal{B}(\mathrm{E})(U_i)^2 + \mathcal{C}(\mathrm{E})(U_i)^2 = 0, \tag{3}$$

for $i = 0,1,\ldots,N$, where $U_i(t) \approx u(x_i,t)$, E is the shift operator, i.e., $\mathrm{E}U_i = U_{i+1}$ and

$$\mathcal{A}(\mathrm{E}) = \frac{\mathrm{E}^{-2} + 26\mathrm{E}^{-1} + 66 + 26\mathrm{E}^{1} + \mathrm{E}^{2}}{120},$$
$$\mathcal{B}(\mathrm{E}) = \frac{-\mathrm{E}^{-2} - 10\mathrm{E}^{-1} + 10\,\mathrm{E}^{1} + \mathrm{E}^{2}}{24\Delta x},$$
$$\mathcal{C}(\mathrm{E}) = \frac{-\mathrm{E}^{-2} + 2\mathrm{E}^{-1} - 2\,\mathrm{E}^{1} + \mathrm{E}^{2}}{2\Delta x^3}.$$

Method (3) is fourth-order accurate in space for regular enough solutions ($u(x,t) \in C^7$), since its truncation error terms are given by:

$$\mathrm{TET}\{u(x,t)\} = \frac{\Delta x^4}{240}\frac{\partial^7 u^2}{\partial x^7} + O(\Delta x^6).$$

However, in solutions of the $K(2,2)$ equation with multiple colliding compactons, shocks (or nonsmooth solutions) are developed reducing the effective order of accuracy and introducing numerical instabilities which may blow up the solution [7,14]. In order to avoid these instabilities, artificial viscosity must be introduced into the non-dissipative method given by Eq. (3). Here, as in Refs. [7,12,14,19], the term $\mu\partial^4 u/\partial x^4$, with μ small enough, is introduced into the left-hand side of Eq. (1) and numerically discretized by means of a second-order accurate five-point difference formula, given by:

$$\mathcal{D}(\mathrm{E})U_i = \frac{\mathrm{E}^{-2} - 4\mathrm{E}^{-1} + 6 - 4\,\mathrm{E}^{1} + \mathrm{E}^{2}}{\Delta x^4}U_i. \tag{4}$$

The main goal of this paper is the comparison of the following four methods for the integration in time of Eq. (3), including the term (4).

- **Method 1.** The application of the first-order, implicit Euler method to Eq. (3) yields