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Two-periodic waves and asymptotic property for generalized 2D Toda lattice equation

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ABSTRACT

In this paper, two-periodic wave solutions are constructed for the (2 + 1)-dimensional generalized Toda lattice equation by using Hirota bilinear method and Riemann theta function. At the same time, we analyze in details asymptotic properties of the two-periodic wave solutions and give their asymptotic relations between the periodic wave solutions and the soliton solutions.

1. Introduction

It is well-known that the Hirota bilinear method is a powerful tool to construct exact solutions of nonlinear equations such as the KdV, NLS, KP, sine–Gordon equations. By applying the bilinear method, people obtained a series of exact solutions to many continuous and discrete nonlinear equations which include Jacobi elliptic periodic solutions, position solutions, complexiton solutions, Casoratian solutions, ghoston solutions and so on [1–12]. However, little work has been done on the periodic wave solutions by using the Hirota method. In fact, the appeal and success of this method lies in the fact that we avoid complicated algebra-geometric theory to directly get quasi-periodic wave solutions. Moreover, it is shown that all parameters appearing in the periodic wave solutions are conditionally free variables, whereas usual quasi-periodic solutions involve some Riemann constants which are difficult to be determined explicitly. Nakamura proposed a convenient way to construct a kind of quasi-periodic solutions of nonlinear equation in his two serial papers [13,14], where the periodic wave solutions of the KdV equation and the Boussinesq equation were obtained by means of the Hirota's bilinear method. Prof. Fan have applied this method to investigate ANNV and the discrete Toda lattice equations to obtain one-periodic wave solutions [15,16]. Very recently, Prof. Ma and his collaborators have extended Nakamura's idea and have given a systematic approach, which we can apply to other nonlinear equations, to get periodic wave solutions [17]. This method not only conveniently obtains quasi-periodic solutions of nonlinear equations, but also directly gives the explicit relations among frequencies, wave-numbers, phase shifts and amplitudes of the waves.

In present paper, motivated by the work of Nakamura, Ma and Fan, we consider the following (2 + 1)-dimensional generalized Toda lattice equation

$$\left[\ln(1+\alpha\partial_t^{-1}(u_{nx})+u_n)\right]_{tx} = u_{n+1}-2u_n+u_{n-1}, \tag{1.1}$$

where $u_n = u(t,x,n)$ is a potential function, α is a free constant. Obviously, Eq. (1.1) can be reduced into well-known 2D Toda lattice equation when $\alpha = 0$. Recently, for the 2D Toda lattice equation, many papers have been focusing their topics on the various exact solutions which include the Gram-type determinant solutions, Gram-type Pfaffian solutions, and ghoston solutions by means of the Wronskian and Pfaffian and Casoratian technique [18–20]. And various Casoratioan type solutions are

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generated for the 2D Toda lattice equation, through Wronskian formulations and Bäcklund transformation[18]. For 1D Toda lattice equation, one-periodic wave solutions are constructed by bilinear method and Riemann theta function [15]. However the one-periodic waves are well-known cnoidal waves, their surface pattern is one-dimensional; However the water surface is two-dimensional which drive us to research multi-dimensional generalization of cnoidal waves. Thus in this paper, we are interested in constructing the two-periodic wave solutions of the Eq. (1.1) and discussing their asymptotic properties.

This paper is organized as follows. In Section 2, we briefly introduce some useful bilinear forms involving the Riemann theta functions. In Section 3, Two-periodic wave solutions are constructed to the Eq. (1.1) by applying the Hirota's bilinear method. Finally we give the asymptotic properties of the two-periodic wave solutions.

2. The bilinear form and the Riemann theta function

In this section, we briefly introduce a useful bilinear form of the Eq. (1.1) and some main points on the Riemann theta function. By the dependent variable transformation [19,20]

$$u_n = \partial_{tx}^2 \ln f_n(t,x,n),$$

the Eq. (1.1) is then transformed into a bilinear form

$$\left(D_xD_t + \alpha D_x^2 - 2e^{D_n} + 2\right) f_n(t,x,n)\cdot f_n(t,x,n) = 0, \tag{2.1}$$

where the bilinear differential operators D_x, D_t and e^{D_n} are defined by

$$D_x^m D_t^n f_n(t,x,n)\cdot g_n(t,x,n) = (\partial_x - \partial_{x'})^m(\partial_t - \partial_{t'})^n f_n(t,x,n) g_n(t',x',n)|_{x'=x,t'=t},$$
$$e^{D_n} f_n \cdot g_n = f_{n+1}g_{n-1}, \quad e^{-D_n} f_n\cdot g_n = f_{n-1}g_{n+1}.$$

The bilinear operators have a good property when acting on exponential functions, namely,

$$D_x^m D_t^n e^{\zeta_1}\cdot e^{\zeta_2} = (\rho_1-\rho_2)^m(\omega_1-\omega_2)^n e^{\zeta_1+\zeta_2},$$
$$e^{D_n}(e^{\zeta_1}\cdot e^{\zeta_2}) = e^{\nu_1-\nu_2}e^{\zeta_1+\zeta_2},$$

where $\zeta_j = \eta_j x + \omega_j t + \nu_j n + \gamma_j$, $j = 1, 2$. More general, we have

$$G(D_x, D_t, e^{D_n}) e^{\zeta_1}\cdot e^{\zeta_2} = G(\eta_1-\eta_2, \omega_1-\omega_2, e^{\nu_1-\nu_2}) e^{\zeta_1+\xi_2}, \tag{2.2}$$

where $G(D_x, D_t, e^{D_n})$ is a polynomial about D_x, D_t and e^{D_n}. This property will be used later and plays a key role in the construction of the periodic wave solutions. Following the Hirota bilinear theory, the Eq. (1.1) admits one-soliton solution

$$u_1 = \partial_{tx}\ln(1+e^{\eta x+\omega t+\nu n+\gamma}), \tag{2.3}$$

where $\omega = \frac{4\sinh^2\frac{\nu}{2}}{\eta} - \alpha\eta$, and η, ν, γ being constants, and two-soliton solution

$$u_2 = \partial_{tx}\ln(1+e^{\zeta_1}+e^{\zeta_2}+e^{\zeta_1+\zeta_2+A_{12}}), \quad e^{A_{12}} = \frac{\sinh^2\frac{\nu_1-\nu_2}{4}}{\sinh^2\frac{\nu_1+\nu_2}{4}}, \tag{2.4}$$

with

$$\zeta_j = \eta_j x + \omega_j t + \nu_j n + \gamma_j, \quad \omega_j = \frac{4\sinh^2\frac{\nu_j}{2}}{\eta_j} - \alpha\eta_j \quad j = 1, 2,$$

where η_j, ν_j, γ_j, $j = 1, 2$ are free constants.

In order to apply the Hirota bilinear method to construct two-periodic wave solutions of the Eq. (1.1), we consider a slightly generalized form of the bilinear Eq. (2.1). Here we look for its solution of the Eq. (1.1) in the form

$$u = u_0 + \partial_{tx}^2 \ln\vartheta(\xi), \tag{2.5}$$

where u_0 is a free constant and is taken as a seed solution of the Eq. (1.1), and phase variable ξ is taken as the form $\xi = (\xi_1,\ldots,\xi_N)^T$, $\xi_j = \rho_j x + \omega_j t + \mu_j n + \delta_j$, $j = 1, 2,\ldots, N$.

By substituting (2.5) into (1.1), we then get the following bilinear form

$$G(D_x, D_t, e^{D_n})\vartheta(\xi)\cdot\vartheta(\xi) = (D_tD_x + \alpha D_x^2 - 2ce^{D_n} + 2 + 2u_0)\vartheta(\xi)\cdot\vartheta(\xi) = 0, \tag{2.6}$$

where $c = c(n)$ is an integration constant. For the bilinear Eq. (2.6), we are interested in its multi-periodic solutions in terms of the Riemann theta function

$$\vartheta(\xi) = \vartheta(\xi,\tau) = \sum_{n\in Z^N} e^{-\pi<\tau n,n>+2\pi i<\xi,n>}. \tag{2.7}$$

Here the integer value vector $n = (n_1,\ldots,n_N)^T \in Z^N$, and complex phase variables $\xi = (\xi_1,\ldots,\xi_N)^T \in C^N$; Moreover, for two vectors $f = (f_1,\ldots,f_N)^T$ and $g = (g_1,\ldots,g_N)^T$, their inner product is defined by