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Fault-tolerant edge and vertex pancyclicity in alternating group graphs

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ABSTRACT

In [J.-M. Chang, J.-S. Yang. Fault-tolerant cycle-embedding in alternating group graphs, Appl. Math. Comput. 197 (2008) 760–767] the authors claim that every alternating group graph AG_n is $(n-4)$-fault-tolerant edge 4-pancyclic. Which means that if the number of faults $|F| \leqslant n-4$, then every edge in $AG_n - F$ is contained in a cycle of length ℓ, for every $4 \leqslant \ell \leqslant n!/2 - |F|$. They also claim that AG_n is $(n-3)$-fault-tolerant vertex pancyclic. Which means that if $|F| \leqslant n-3$, then every vertex in $AG_n - F$ is contained in a cycle of length ℓ, for every $3 \leqslant \ell \leqslant n!/2 - |F|$. Their proofs are not complete. They left a few important things unexplained. In this paper we fulfill these gaps and present another proofs that AG_n is $(n-4)$-fault-tolerant edge 4-pancyclic and $(n-3)$-fault-tolerant vertex pancyclic.

1. Introduction

An alternating group graph AG_n, $n \geqslant 3$, was introduced by Jwo et al. [3] as an interconnection network topology for computing systems. The graph AG_n has vertices labeled by even permutations of the set $\{1,\ldots,n\}$. Two vertices p and q are neighbors if one of them is obtained from the other by rotating three symbols: the first, second and ith, for some $i \geqslant 3$. The graph AG_3 has three vertices 123, 231 and 312, every two are connected. AG_4 is presented in Fig. 1. The n dimensional alternating group graph AG_n is a vertex symmetric and edge symmetric regular graph with $n!/2$ vertices, $n!\,(n-2)/2$ edges, vertex degree $2n-4$ and diameter $\lfloor 3n/2 \rfloor - 3$ (see [3]). By F we shall denote the set of faulty vertices. In [1] the authors claim that AG_n is $(n-4)$-fault-tolerant edge 4-pancyclic. Which means that if the number of faults $|F| \leqslant n-4$, then every edge in $AG_n - F$ is contained in a cycle of length ℓ, for every $4 \leqslant \ell \leqslant n!/2 - |F|$. They also claim that AG_n is $(n-3)$-fault-tolerant vertex pancyclic. Which means that if $|F| \leqslant n-3$, then every vertex in $AG_n - F$ is contained in a cycle of length ℓ, for every $3 \leqslant \ell \leqslant n!/2 - |F|$. Their proofs are not complete. They left a few important things unexplained:

(1) When proving, by induction, Theorem 1 (that AG_n is $(n-4)$-fault-tolerant edge 4-pancyclic) they decompose AG_n into subgraphs $A^1,\ldots,A^n$. By induction hypothesis, shortest cycles are in these subgraphs. To obtain longer cycles they take a cycle C already build and extend it into the next subgraphs using so called 4-cycle structures. If the cycle C is contained in one subgraph A^i, then it is easy to see that C can be extended into a new subgraph A^j. But the authors do not explain how to find such extending structure if C goes through more than one subgraph and there are only few subgraphs unvisited.
(2) The arguments used in Case 2 of the proof of Theorem 1 do not work for $n = 5$. It is not possible to repartition AG_5 with $|F| = 1$ fault into subgraphs with 0 faults each.
(3) In the proof of Theorem 2 (that AG_n is $(n-3)$-fault-tolerant vertex pancyclic) in Case 1 it is not explained why subgraph H^k with $f_k = n-4$ faults is edge 4-pancyclic. Theorem 1 works if $f_k \leqslant (n-1)-4 = n-5$.

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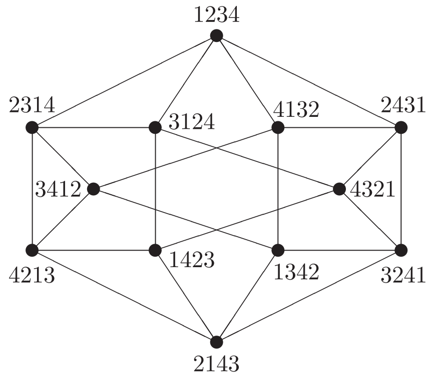

Fig. 1. AG_4

In this paper we fulfill the gaps and present another proofs that AG_n, $n \geqslant 4$, is $(n-4)$-fault-tolerant edge 4-pancyclic and $(n-3)$-fault-tolerant vertex pancyclic. In [1] the authors showed that both these bounds are optimal.

2. Alternating group graph

An alternating group graph AG_n, $n \geqslant 3$, has vertices labeled by even permutations of the set $\{1,\ldots,n\}$. The permutation $p = (p_1,\ldots,p_n)$ is even if it contains the even number of inversions. The inversion is a pair of numbers i, j, $1 \leqslant i < j \leqslant n$ such that $p_i > p_j$. For every i, $3 \leqslant i \leqslant n$, let $g_i^+ = (12i)$ be the permutation which rotates symbols in positions 1, 2 and i from right to left; and $g_i^- = (1i2)$ be the permutation which rotates these symbols from left to right. Two vertices p and q are connected by an edge if and only if $q = pg_i^+$ or $q = pg_i^-$, for some $i \geqslant 3$. For example in AG_4 the vertex $p = 1234$ is connected with $pg_3^+ = 2314$ and $pg_3^- = 3124$. Observe that if $q = pg_i^+$ then $p = qg_i^-$. The n dimensional alternating group graph AG_n is a vertex symmetric and edge symmetric regular graph with $n!/2$ vertices, $n!\,(n-2)/2$ edges, vertex degree $2n-4$ and diameter $\lfloor 3n/2 \rfloor - 3$ (see [3]).

The graph AG_n can be divided into subgraphs $A^1, \ldots, A^n$, each A^i contains vertices with i on the last symbol. The subgraph A^i is isomorphic with AG_{n-1}. We can also divide AG_n according to other position, say k, for some $3 \leqslant k \leqslant n-1$. Then A^i contains vertices with i on the kth position. Note that every two vertices u and v must differ in some symbol $k \geqslant 3$ and we can decompose AG_n in such a way that u and v are in different subgraphs and we can always assume that faulty vertices are not in one subgraph (if there are more than one fault). On the other hand we can also divide AG_n, $n \geqslant 4$, in such a way that two ends of an edge are in one subgraph. This is because they differ only in one position $i \geqslant 3$.

Every vertex $u \in A^i$ is connected with exactly two vertices u' and u'' outside A^i (they are in two different subgraphs). We will call the edges (u,u') and (u,u'') external edges. Other edges we shall call internal. For each internal edge $(u,v) \in A^i$ with $u = (kj\ldots i)$ and $v = (jk'\ldots i)$, there exist adjacent vertices $s = (ik\ldots j)$ and $t = (k'i\ldots j)$ both in A^j which form the 4-cycle (u,s,t,v). We shall say that the edge (u,v) is of color j or that it is connected (by a 4-cycle) with the edge (s,t) in A^j. If a subgraph A^i is of dimension 4 and is isomorphic to AG_4 (see Fig. 1) then there are 4 colors, the edges of each color form a cycle of length 6. For example, the cycle 1234, 4132, 1342, 2143, 1423, 3124 contains edges of color 1. If a subgraph A^i is of dimension 5, then it can be divided (according to the 5th position) into 5 subgraphs $A_1^i, A_2^i, A_3^i, A_4^i, A_5^i$. Each of A_j^i is isomorphic to AG_4, and contains 4 colors (all colors except i and j) and edges of each color in A_j^i form a cycle of length 6. Similarly for higher dimensions. AG_n can be divided into n subgraphs $A^1, \ldots, A^n$ according to the last position. Each A^i can be divided into $(n-1)$ subgraphs $A_1^i, \ldots, A_{n-1}^i$ according to the last by one position and so on. But the color of the edge depends only on the first two symbols and is the same in each subgraph. Moreover if an edge (u,v) is in the subgraph A_k^i and is connected with the edge (u',v') in A^j, then the edge (u',v') is in the subgraph A_k^j. There are $(n-2)!$ external edges joining two different subgraphs A^i and A^j. If x and y are two vertices in A^i, then external edges (x,x') and (y,y') cannot meet in one vertex. Otherwise external edges from $x' = y'$ would go to one subgraph A^i, which is impossible. It is easy to see that we can choose external edges (x,x') and (y,y') in such a way that x' and y' are in two different subgraphs. By F we shall denote the set of faulty vertices; $f_i = |A^i \cap F|$ denotes the number of faulty vertices in A^i and $h_i = |A^i - F|$ denotes the number of healthy vertices in A^i.

Lemma 1. *For any two edges $e, f \in AG_4$, there exists a Hamiltonian cycle going through e and f.*

Proof. By symmetry of AG_4 we can assume that the edge $e = (1234, 2314)$. Consider three Hamiltonian cycles:

$$(1234, 2314, 4213, 3412, \quad 4132, 2431, 3241, 1342, \quad 2143, 1423, 4321, 3124),$$
$$(1234, 2314, 3124, 4321, \quad 2431, 3241, 2143, 1423, \quad 4213, 3412, 1342, 4132),$$
$$(1234, 2314, 3412, 4213, \quad 2143, 1423, 3124, 4321, \quad 3241, 1342, 4132, 2431).$$