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Representations for the Drazin inverses of 2×2 block matrices ☆

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ABSTRACT

Let M denote a 2×2 block complex matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A and D are square matrices, not necessarily with the same orders. In this paper explicit representations for the Drazin inverse of M are presented under the condition that $BD^iC = 0$ for $i = 0, 1, \ldots, n-1$, where n is the order of D.

1. Introduction

For a square matrix $A \in \mathbb{C}^{n\times n}$, the Drazin inverse of A is a matrix $A^D \in \mathbb{C}^{n\times n}$ satisfying

$$AA^D = A^DA, \quad A^DAA^D = A^D, \quad A^{k+1}A^D = A^k$$

for some nonnegative integer k. It is well-known that A^D always exists and is unique. If $A^{k+1}A^D = A^k$ for some nonnegative integer k, then so does for all nonnegative integer $l \geqslant k$, and the smallest nonnegative integer such that the equation holds is equal to ind(A), the index of A, which is defined to be the smallest nonnegative integer such that rank (A^{k+1}) = rank (A^k). We adopt the convention that $A^0 = I_n$, the identity matrix of order n, even if $A = 0$, and the index of the zero matrix is defined to be 1. We write $A^\pi = I - AA^D$. For more details we refer the reader to [1,4].

The Drazin inverse is first studied by Drazin [23] in associative rings and semigroups. The Drazin inverse of complex matrices and its applications are very important in various applied mathematical fields like singular differential equations, singular difference equations, Markov chains, iterative methods and so on [4,5,24,31,35,37].

The study on representations for the Drazin inverse of block matrices essentially originated from finding the general expressions for the solutions to singular systems of differential equations [3–5], and then stimulated by a problem formulated by Campbell [5]: establish an explicit representation for the Drazin inverse of 2×2 block matrices $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ in terms of the blocks of the partition, where the blocks A and D are assumed to be square matrices. For a deeper discussion of applications of the Drazin inverse of a 2×2 block matrix, we refer the reader to [4,37]. Until now, there has been no explicit formula for the Drazin inverse of general 2×2 block matrices. Meyer and Rose [32], and independently Hartwig and Shoaf [26], first gave the formulas for block triangular matrices, and since then many less restrictive assumptions are considered [2,7,9,11–14,20,22,25,26,29,32–34,36], for example,

(1) $BC = 0$, $BD = 0$ and $DC = 0$ (see [20]);
(2) $BC = 0$, $DC = 0$ (or $BD = 0$) and D is nilpotent (see [25]);

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(3) $BC = 0$ and $DC = 0$ (see [14]);
(4) $BC = 0$, $BDC = 0$ and $BD^2 = 0$ (see [22]);
(5) $BD^{\pi}C = 0$, $BDD^D = 0$ and $DD^{\pi}C = 0$ (see [22]).

Related topics are to find representations for the Drazin inverse and the generalized Drazin inverse of the sum of two matrices [6,27,28,34] and operator matrices on Banach spaces [8,10,15–19,21].

It is clear that the condition in (4) above implies that $BD^iC = 0$ for any nonnegative integer i, by which our work is motivated. Though this condition looks like a restrictive one, it is really weaker than many ones in the literature, especially those in (1)–(5) and [16] which considers the generalized Drazin inverses, as shown in Examples 4.1 and 4.2, respectively.

In this paper, explicit expressions for the Drazin inverse of the 2×2 block matrix M are provided under the condition that $BD^iC = 0$ for $i = 0, 1, \ldots, n-1$, where n is the order of D, from which many results are unified and many formulas can be derived, especially those in [14,20,22,25].

For notational convenience, we define a sum to be 0, whenever its lower limit is bigger than its upper limit.

2. Preliminary

Lemma 2.1. *For $D \in \mathbb{C}^{n\times n}$ and matrices B, C of appropriate orders, if $BD^iC = 0$ for $i = 0, 1, \ldots, n-1$, then $BD^kC = 0$ and $B(D^D)^kC = 0$ for any nonnegative integer k.*

Proof. Let $f(\lambda) = \lambda^n - a_1\lambda^{n-1} - \cdots - a_n$ be the characteristic polynomial of D. By the Cayley–Hamilton theorem, $f(D) = 0$. Thus

$$D^n = a_1D^{n-1} + \cdots + a_nI,$$

from which an induction on k yields $BD^kC = 0$ for any nonnegative integer k. Since D^D is expressible as a polynomial of D (see [1]), we have $B(D^D)^kC = 0$ for any nonnegative integer k. $\square$

For simplicity of notation, we adopt the notation $A^k(\varepsilon) = (A^k + \varepsilon I)^{-1}$ for any positive integer k, where ε is a sufficiently small positive real number such that $(A^k + \varepsilon I)^{-1}$ exists.

Lemma 2.2 [30]. *For $A \in \mathbb{C}^{n\times n}$, the following statements are equivalent*:
(1) $\mathrm{ind}(A) = k$;
(2) *k is the smallest nonnegative integer for which the limit $\lim_{\varepsilon\to 0}\varepsilon^k A(\varepsilon)$ exists.*

Furthermore, for every nonnegative integer p, if $p \geqslant \mathrm{ind}(A)$ then

$$A^D = \lim_{\varepsilon\to 0} A^{p+1}(\varepsilon)A^p.$$

Since all the limits in this paper are taken as $\varepsilon \to 0$, we shall simply write "lim" instead of "$\lim_{\varepsilon\to 0}$".

3. Main results

Let M denote a 2×2 block complex matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ satisfying the following conditions:

$$A \in \mathbb{C}^{m\times m}, \quad D \in \mathbb{C}^{n\times n} \quad \text{and} \quad BD^iC = 0, \quad \text{for } i = 0, 1, \ldots, n-1. \tag{3.1}$$

Then for any nonnegative integer k, a calculation yields

$$M^k = \begin{pmatrix} A^k & B_k \\ C_k & D^k + N_k \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} B_{k+1} &= B_kD + A^kB = AB_k + BD^k, \\ C_{k+1} &= C_kA + D^kC = DC_k + CA^k, \\ N_{k+1} &= C_kB + N_kD = DN_k + CB_k. \end{aligned}$$