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Potential method applied to Boussinesq equation

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ABSTRACT

In this paper a solution of the nonlinear Boussinesq equation is presented using the potential similarity transformation method. The equation is first written in a conserved form, a potential function is then assumed reducing it to a system of equations which is further solved through the group transformation method. New transformations are found.

1. Introduction

Boussinesq in 1872 [1] described the propagation of solitary waves with small amplitude on the surface of shallow water in the form

$$u_{tt} + uu_{xx} + (u_x)^2 + u_{xxxx} = 0, \tag{1}$$

or

$$u_{tt} + (uu_x)_x + u_{xxxx} = 0. \tag{2}$$

These equations attract the attention of searchers due to the travelling property of its solution and its wide range of applications in physics like sound waves in laser beams, electromagnetic waves interacting with transversal optical phonons in nonlinear dielectrics [2], magneto sound waves in plasmas [3]. Eq. (1) was solved by Clarkson and Kruskal [4] using the direct method and lead to new similarity transformations. Levi et al. [5] made the link between the direct method and Lie transformation introducing conditional symmetries. Clarkson et al. [6] non-classical similarity transformation of the generalized Boussinesq equation, lead to a two-soliton solution. Integrable boundary conditions for Boussinesq equation were deduced by Vil'danov [7] through the Lax pair representation of the problem and particular solutions of the boundary problem were found. In 2007 Yan [8] solved a family of higher-dimensional generalized Boussinesq equations using an independent variable form; $\eta = kx + W(y,t)$. The problem was then solved and reduced to two nonlinear ordinary differential equations whose exact solutions were given for specific forms of $W(y,t)$. In 2008 Wang et al. [9] investigated the existence and uniqueness of the solution to the Cauchy problem for a class of Boussinesq equation, while Wang [10] in 2009 proved the existence and the uniqueness of a global solution for the generalized Boussinesq Cauchy problem and found that the wave amplitude $u(x,t)$ decays to zero as 't' tends to the infinity. Finally Bruzon et al. [11] presented a full analysis of a family of Boussinesq equations and proved that the non-classical method applied to these equations leads to new symmetries, which cannot be obtained by Lie classical method.

We here introduce the potential method which consists in the reduction of (2) to a system of equations through the use of an auxiliary function. This system is then solved using the group transformation method [12–14].

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2. Mathematical formulation

Consider the Boussinesq equation

$$\frac{\partial^2 u^*}{\partial t^2} + \left(u^* \frac{\partial u^*}{\partial x}\right)_x + \frac{\partial^4 u^*}{\partial x^4} = 0, \tag{3}$$

subjected to initial conditions

$$u^*(x,0) = f(x), \tag{4.a}$$

$$\frac{\partial u^*(x,0)}{\partial t} = 0, \tag{4.b}$$

this equation is written in a conserved form

$$\left(\frac{\partial u^*}{\partial t}\right)_t = -\left(\frac{(u^*)^2}{2} + \frac{\partial^2 u^*}{\partial x^2}\right)_{xx}. \tag{5}$$

The corresponding potential system is written as

$$\frac{\partial^2 w}{\partial x^2} = \frac{\partial u^*}{\partial t}, \tag{6}$$

$$\frac{\partial w}{\partial t} = -\left[\frac{(u^*)^2}{2} + \frac{\partial^2 u^*}{\partial x^2}\right], \tag{7}$$

where $w(x,t)$ is an auxiliary function. A normalization of the initial condition (4.a) is obtained by setting

$$u(x,t) = \frac{u^*(x,t)}{f(x)}, \tag{8}$$

hence (6) and (7) reduce to

$$w_{xx} = u_t f, \tag{9}$$

$$w_t = \frac{-1}{2} u^2 f^2 - u_{xx} f - 2u_x f_x - u f_{xx}. \tag{10}$$

This differential system is subjected to the initial conditions

$$u(x,0) = 1, \tag{11}$$

$$u_t(x,0) = 0. \tag{12}$$

2.1. Group formulation

A one-parameter group of the form

$$\bar{S} = C^S(a) S + K^S(a), \tag{13}$$

is used to reduce the system of partial equations to a system of ordinary differential equations. S and $\bar{S}$ refers to variables $(x,t;u,w,f)$ before and after transformation. C^S, K^S are coefficients function of the group unity parameter 'a'. First and second order partial derivative are expressed as

$$\bar{S}_{\bar{i}} = \left(\frac{C^s}{C^i}\right) S_i, \tag{14}$$

$$\bar{S}_{\bar{i}\bar{j}} = \left(\frac{C^s}{C^i C^j}\right) S_{ij}, \tag{15}$$

where (i,j) subscripts refer to the variables (x,t).

2.2. Transformation of the system of potential equations

The system of Eqs. (9) and (10) is invariantly transformed as follows:

$$\bar{w}_{\bar{x}\bar{x}} - \bar{u}_{\bar{t}} \bar{f} = H_1(a)[w_{xx} - u_t f], \tag{16}$$

$$\bar{w}_{\bar{t}} + \frac{1}{2} \bar{u}^2 \bar{f}^2 + \bar{u}_{\bar{x}\bar{x}} \bar{f} + 2\bar{u}_{\bar{x}} \bar{f}_{\bar{x}} + \bar{u} \bar{f}_{\bar{x}\bar{x}} = H_2(a)\left[w_t + \frac{1}{2} u^2 f^2 + u_{xx} f + 2u_x f_x + u f_{xx}\right], \tag{17}$$