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Analytical and numerical solutions of a one-dimensional fractional-in-space diffusion equation in a composite medium ☆

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ABSTRACT

In this paper a one-dimensional space fractional diffusion equation in a composite medium consisting of two layers in contact is studied both analytically and numerically. Since domain decomposition is the only approach available to solve this problem, we at first investigate analytical and numerical strategies for a composite medium with the same fractal dimension in each layer to ascertain which domain decomposition approach is the most accurate and consistent with a global solution methodology, which is available in this case. We utilise a matrix representation of the fractional-in-space operator to generate a system of linear ODEs with the matrix raised to the same fractional exponent. We show that the global and domain decomposition numerical strategies for this problem produce simulation results that are in good agreement with their analytic counterparts and conclude that the domain decomposition that imposes the Neumann condition at the interface produces the most consistent results. Finally, we carry this finding to study the composite problem with different fractal dimensions, where we again favourably compare analytic and numerical solutions. The resulting method can be naturally extended to space fractional diffusion in a composite medium consisting of more than two layers.

1. Introduction

Recently a growing number of researchers have begun to utilise fractional calculus for some important applications in the physical sciences and this interest has resulted in fractional differential equations being now widely accepted across many fields of science and engineering [1,2,7,8,13,29,33–36,38,44]. Data in many fields of applications display multifractal scaling, such as for example multifractional Brownian motion [6,4]. A variety of multiplicative cascades and iterated function systems has been shown to generate multifractals [30,21,12,11,15]. Brownian motion in multifractal time and most Lévy processes are also known to have multifractal paths [20,40]. Some illustrative examples of application of multifractal analysis includes Refs. [14,31,41,3].

Markov processes associated with pseudodifferential operators with smooth symbols were studied by Bass [5], Jacob and Leopold [19], Jacob [18], Komatsu [23] and Kikuchi and Negoro [22], for example. In particular, Bass [5] considered the generator $-(-\Delta)^{\alpha(x)/2}$, where Δ is the Laplacian, for a function $\alpha(x)$ satisfying $0 < \alpha(x) < 2$ and called the generated process an isotropic stable-like process. Komatsu [23] extended this class to generalized stable-like processes. Jacob and Leopold [19] showed that there exists a Feller semigroup generated by the pseudodifferential operator whose symbol is the function

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$-(1+|\xi|^2)^{\alpha(x)},\ 0<\inf\alpha(x)\leqslant\sup\alpha(x)\leqslant 2$. Kikuchi and Negoro [22] extended the result to strongly elliptic pseudodifferential operators with suitable variable order. Ruiz-Medina et al. [43] introduced a class of Markov processes whose transition probability densities are defined by multifractional pseudodifferential equations on compact domains with variable local dimension. The infinitesimal generators of these Markov processes are given by the trace of strongly elliptic pseudodifferential operators of variable order on such domains. The results derived provide an extension of some existing classes of multifractional Markov processes. In particular, pseudostable processes are defined on domains with variable local dimension in this framework.

Numerical analysis of multifractional processes is still at an early stage of development. A particular situation that we are interested in concerns fractional diffusion in a composite medium consisting of several layers of different fractal dimensions. This is an example of (multifractional) processes with variable singularity order. This paper aims to investigate appropriate analytical and numerical strategies for such a situation.

Some numerical methods for solving the space fractional partial differential equations have been proposed recently. Liu et al. [25,26] transformed the space fractional partial differential equation into a system of ordinary differential equations (method of lines), which was then solved using backward differentiation formulae. Roop [42] investigated the numerical approximation of the variational solution to the space fractional advection dispersion equation. Meerschaert et al. [32] examined finite difference approximations for space fractional advection–dispersion flow equations. Shen et al. [45] proposed an explicit finite difference approximation for the space fractional diffusion equation and gave a supporting error analysis. Liu et al. [28] discussed an approximation of the Lévy–Feller advection–dispersion process by a random walk and finite difference method. Liu et al. [27] also discussed the stability and convergence of the difference methods for the space–time fractional advection–diffusion equation.

In previous research by the authors [16,17] fractional-in-space diffusion equations have been studied both analytically and numerically. Here, a new matrix transfer technique for solving the fraction-in-space diffusion equation was proposed, which is based on using a standard discretisation of the fractional-in-space operator to generate a system of linear ODEs with the matrix raised to the same fractional exponent. In this study we consider the diffusion process in a composite medium consisting of several layers in contact with different fractional exponents. For easier exposition, the theory is illustrated for two slabs $0\leqslant x<l,\ l\leqslant x\leqslant L$ with exponents α_1 and α_2, respectively. Specifically, an approximate solution is sought for the following problem.

Problem 1. Solve the following initial-boundary value problem (BVP) in one-dimension:

$$\frac{\partial\varphi_1}{\partial t}=-\kappa_1(-\nabla^2)^{\frac{\alpha_1}{2}}\varphi_1,\quad 0<x<l,$$
$$\frac{\partial\varphi_2}{\partial t}=-\kappa_2(-\nabla^2)^{\frac{\alpha_2}{2}}\varphi_2,\quad l<x<L$$

with the initial condition

$$\varphi_1(x,0)=F_1(x),\quad 0<x<l;\quad \varphi_2(x,0)=F_2(x),\quad l<x<L,$$

together with one of the following boundary conditions for $t\geqslant 0$:

(i) $\varphi_1(0,t)=f(t),\ \varphi_2(L,t)=g(t)\ \ldots\ (BC)_1$;
(ii) $\frac{\partial\varphi_1}{\partial x}(0,t)=f(t),\ \frac{\partial\varphi_2}{\partial x}(0,t)=g(t)\ \ldots\ (BC)_2$;
(iii) $\frac{\partial\varphi_1}{\partial x}(0,t)+\beta\varphi_1(0,t)=f(t),\ \frac{\partial\varphi_2}{\partial x}(0,t)+\beta\varphi_2(0,t)=g(t)\ \ldots\ (BC)_3$

and the interfacial conditions (transmission coefficients) for $t\geqslant 0$:

(i) $\varphi_1(l,t)=\varphi_2(l,t)$,
(ii) $K_1\frac{\partial\varphi_1}{\partial x}(l,t)=K_2\frac{\partial\varphi_2}{\partial x}(l,t)$.

Here κ_i is the thermal diffusivity and K_i is the thermal conductivity, which can be different in each layer $i(=1,2)$.

In [16,17], we discussed both the analytic and numerical solutions to space fractional diffusion equations (SFDE) of the following type.

Problem 2. Solve the following initial-boundary value problem (BVP):

$$\frac{\partial\varphi}{\partial t}=-\kappa(-\nabla^2)^{\frac{\alpha}{2}}\varphi+g,\quad \text{on }\Omega,\tag{1.1}$$

with the boundary conditions (B.Cs):

$$B(\varphi)=f,\quad \text{on }\partial\Omega,$$

and the initial condition (I.C):