Applied Mathematics and Computation 216 (2010) 1707–1714

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Generalizations of the nonstationary multisplitting iterative method for symmetric positive definite linear systems

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A R T I C L E I N F O

Keywords:
Nonstationary multisplitting
Iterative method
Convergence
Symmetric positive definite linear systems

A B S T R A C T

In this paper, we generalize the nonstationary parallel multisplitting iterative method for solving the symmetric positive definite linear systems. With several choices of variable weighting matrices, the convergence properties of these generalized methods can be improved. Finally, the numerical comparison of several nonstationary parallel multisplitting methods are shown.

1. Introduction

In order to get an iterative method to solve the large sparse system of linear equations

$$Ax = b, \quad A \in \mathbb{R}^{n\times n} \text{ nonsingular and } b \in \mathbb{R}^n, \tag{1.1}$$

on a multiprocessor system. O'Leary and White [20] first proposed parallel methods based on multisplitting technique of matrices in 1985. Later, this technique was studied by many authors. By varying combinations of weighting matrices (see [26]), introducing relaxation parameter(s) (see [1,2,12,24]), or constructing inner/outer iterations (see [17,21]), many researchers have developed and generalized both method models and convergence theories of the matrix multisplitting iterations for solving the large sparse linear system (1.1) on the SIMD (Single Instruction stream-Multiple Data stream) multiprocessor system. All these works make the matrix multisplitting technique become more bounteous and complete.

In an efficient implementation of a multisplitting method on a multiprocessor system, to avoid loss of time and efficiency in processor utilization due to the unbalance of the workloads among processors of a multiprocessor system, the asynchronous parallel iterative methods may be preferable to their synchronous alternations. Thus, by a technical combination of both chaotic iteration idea of Chazan and Miranker [10] and matrix multisplitting technique of O'Leary and White [20], Bru et al. [7] proposed two models of parallel multisplitting chaotic iterations for solving the large sparse linear system (1.1), and their work empties into vitality and affords novel ways for studying the asynchronous parallel iterative methods for solving large sparse linear systems in the sense of matrix multisplitting. By applying the accelerated overrelaxation (AOR) technique of Hadjidimos [16] and considering the concrete characteristic of the MIMD (Multiple Instruction stream-Multiple Data stream) multiprocessor system [6,25], further generalized and improved the aforementioned asynchronous iteration models, and established a series of useful asynchronous parallel matrix multisplitting AOR iterative methods for solving the linear system (1.1). In particular, the method that was given by Bai and Wang [5] had great generality, that is, it reduces to the nonstationary multisplitting iterative method described in [9] when $N_i = 0$; it becomes the stationary multisplitting two-stage iterative method studied in [21] when $s(i,k) = s$ $(i = 1,2,\ldots,\alpha; k = 1,2,\ldots)$; it recovers the multisplitting iterative method proposed in [20] when $s(i,k) = 1$ $(i = 1,2,\ldots,\alpha; k = 1,2,\ldots)$ and $N_i = 0$. Moreover, it recovers the two-stage

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0096-3003/$ - see front matter
doi:10.1016/j.amc.2009.12.033

iterative method discussed in [3,13] when $\alpha = 1$. Besides above-mentioned works, we refer the readers to [1–26], and references therein.

In this study, we focus on generalizing the weighting matrices E_i $(i = 1, 2, \ldots, m)$ to $E_i^{(k)}$ $(i = 1, 2, \ldots, m; k = 0, 1, 2, \ldots,)$ for the nonstationary multisplitting method presented in [9], which is also special case that method proposed by Bai and Wang [5]. Thus, $E_i^{(k)}$ enable more approximate to the exact solution for k-step iteration. Similar to [5], we study these generalized methods for solving the symmetric positive definite linear systems. With several choices of variable weighting matrices, in particular, the nonnegative conditions are eliminated, the convergence properties of these generalized methods can be improved. Numerical example is shown that the methods without *nonnegative* variable weighting matrices are much more effective than the Algorithm 2 studied in [9].

The contents of this paper are arranged as follows. We first give some notations and preliminaries in Section 2, and then some generalizations of the parallel multisplitting method are put forward in Section 3, the convergence theories of these generalizations of the parallel multisplitting methods are established in Section 4. Finally, we give the comparison of several parallel nonstationary multisplitting methods by the numerical example.

2. Preliminaries

Here are some essential notations and preliminaries. $\mathbb{R}^{n\times n}$ is used to denote the $n \times n$ real matrix space, and $\mathbb{R}^n$ the n-dimensional real vector space. A^T denotes the transpose of A. Similarly the transpose of a vector $x \in \mathbb{R}^n$ is denoted by x^T.

A matrix $A \in \mathbb{R}^{n\times n}$ is called symmetric positive definite(or semidefinite), if it is symmetric and for all $x \in \mathbb{R}^n$, $x \neq 0$, it holds that $x^TAx > 0 (x^TAx \geqslant 0)$. The spectral radius of the matrix A is denoted by $\rho(A)$. A matrix H is called convergent if $\lim_{k\to\infty} H^k$ exists.

$A = M - N$ is called a splitting of the matrix A if $M \in \mathbb{R}^{n\times n}$ is nonsingular; this splitting is called a convergent splitting if $\rho(M^{-1}N) < 1$; it is called a P-regular splitting of the symmetric positive definite matrix A if $M^T + N$ is positive definite, see [22].

The multisplitting method (see [20]) consists of having a collection of splittings

$$A = M_i - N_i, \quad i = 1, 2, \ldots, m, \quad M_i \text{ nonsingular}, \tag{2.1}$$

and E_i $(i = 1, 2, \ldots, m)$ be m *nonnegative* diagonal weighting matrices such that $\sum_{i=1}^m E_i = I$ (the identity matrix), and the following Algorithm is performed.

Algorithm 2.1 (*Multisplitting*). Given the initial vector $x^{(0)}$.

For $k = 0, 1, 2, \ldots,$ until convergence.
  For $i = 1$ to m
    $M_i y_i = N_i x^{(k)} + b,$
  $x^{(k+1)} = \sum_{i=1}^m E_i y_i.$

As it can be appreciated, Algorithm 2.1 corresponds to the following iteration

$$x^{(k+1)} = Tx^{(k)} + \sum_{i=1}^m E_i M_i^{-1} b, \quad k = 0, 1, 2, \ldots, \tag{2.2}$$

where $T = \sum_{i=1}^m E_i M_i^{-1} N_i$ is the iteration matrix.

Algorithm 2.2 (*Nonstationary Multisplitting (see [9])*). Given the initial vector $x^{(0)}$.

For $k = 0, 1, 2, \ldots,$ until convergence.
  In processor i, $i = 1$ to m
    $y_i^{(0)} = x^{(k)}$
    For $j = 1$ *to* $s(i,k)$
      $M_i y_i^{(j)} = N_i y_i^{(j-1)} + b,$
  $x^{(k+1)} = \sum_{i=1}^m E_i y_i^{(s(i,k))}.$

Lemma 2.3 [18]. *Let A be a symmetric positive definite matrix. Assume the splitting $A = M - N$ is P-regular. Given $s \geqslant 1$, there exists a unique splitting $A = F - G$ such that $(M^{-1}N)^s = F^{-1}G$. Moreover, the splitting is P-regular.*

Lemma 2.4 ([14,18]). *Assume that A is a symmetric positive definite matrix, let $A = F - G$ be P-regular splitting, then there exists a positive number r such that*

$$\|A^{\frac{1}{2}}(F^{-1}G)A^{-\frac{1}{2}}\|_2 \leqslant r < 1.$$