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On parameterized block triangular preconditioners for generalized saddle point problems

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ABSTRACT

In this paper, we discuss two classes of parameterized block triangular preconditioners for the generalized saddle point problems. These preconditioners generalize the common block diagonal and triangular preconditioners. We will give distributions of the eigenvalues of the preconditioned matrix and provide estimates for the interval containing the real eigenvalues. Numerical experiments of a model Stokes problem are presented.

1. Introduction

We consider the solution of the generalized saddle point linear system

$$\begin{bmatrix} A & B^T \\ B & -C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}, \quad \text{or } \mathscr{A}u = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, $B \in \mathbb{R}^{m\times n}$ has full rank, $C \in \mathbb{R}^{m\times m}$ is symmetric and positive semi-definite, and $m \leqslant n$. Problem (1.1) arises in a variety of problems, such as constrained quadratic programming, constrained least-squares problems, mixed finite-element approximations of elliptic PDEs, computational fluid dynamics, and so on. We refer the reader to [14] for a general discussion.

It has been studied that there are n positive and m negative eigenvalues of the coefficient matrix of the system (1.1). For large n and m, it may be attractive to use iterative methods. In particular, Krylov subspace methods might be used. It is often advantageous to use a preconditioner with such iterative methods. The preconditioner should reduce the number of iterations required for convergence but not significantly increase the amount of computation required at each iteration. Preconditioning for system (1.1) has been studied in many papers, such as block diagonal preconditioners [20,24], block triangular preconditioners [4,7,16,22,26–28], constraint preconditioners [7,8,15,17–19], HSS preconditioners [1,3,5,9,13,21], restrictively preconditioned conjugate gradient methods [6,12], matrix splitting preconditioners [23,25] and so on. Recently, Simoncini [22] and Cao [16] studied the application of the block triangular preconditioners

$$\mathscr{P} = \begin{bmatrix} \widehat{A} & B^T \\ 0 & -\widehat{C} \end{bmatrix},$$

and

$$\mathscr{G} = \begin{bmatrix} \widehat{A} & B^T \\ 0 & \widehat{C} \end{bmatrix},$$

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respectively, where $\widehat{A} \in \mathbb{R}^{n\times n}$ and $\widehat{C} \in \mathbb{R}^{m\times m}$ are symmetric and positive definite together with a Krylov subspace iterative solver. It has been studied that many matrix splitting preconditioners are possibly obtained through the simple iterative methods (e.g., Jacobi, symmetric Gauss–Seidel (SGS), successive overrelaxation (SOR), symmetric successive overrelaxation (SSOR) preconditioners [2,10,11,23,25]). The preconditioners $\mathscr{P}$ and $\mathscr{G}$ can be taken from the generalized Gauss–Seidel splitting of the coefficient matrix $\mathscr{A}$. Choosing appropriate ω, SOR method has better convergence rate than the Gauss–Seidel method. This motivates us to study the following two parameterized block triangular preconditioners

$$\mathscr{P}_1 = \begin{bmatrix} \widehat{A} & \omega B^T \\ 0 & -\widehat{C} \end{bmatrix} \quad \text{and} \quad \mathscr{P}_2 = \begin{bmatrix} \widehat{A} & \omega B^T \\ 0 & \widehat{C} \end{bmatrix},$$

where $\widehat{A} \in \mathbb{R}^{n\times n}$ and $\widehat{C} \in \mathbb{R}^{m\times m}$ are symmetric and positive definite, ω is a real parameter and $\omega \geqslant 0$. In the case of $\omega = 0, \mathscr{P}_1$ and $\mathscr{P}_2$ are block diagonal preconditioners. They have been discussed in many papers, see, for example, [20,24]. In contrast with our preconditioners, we only consider the case $\omega = 0$ in the numerical experiment. In the case of $\omega = 1, \mathscr{P}_1$ and $\mathscr{P}_2$ are block triangular preconditioners, which have been studied in [22,16], respectively. It is worth pointing out that in [7], Bai and Ng also proposed the block triangular preconditioners $\mathscr{P}$ and $\mathscr{G}$. Some interesting results were given. Clearly, we give the generalized preconditioners. Simoncini [22] has showed that the preconditioned matrix $\mathscr{A}\mathscr{P}^{-1}$ is positive stable, and the real part of each eigenvalue is contained in a range. Cao [16] has showed that the preconditioned matrix $\mathscr{A}\mathscr{G}^{-1}$ is indefinite with all eigenvalues being real and the estimate for the interval containing these real eigenvalues has been studied. In this paper, we will show that using the preconditioners $\mathscr{P}_1$ and $\mathscr{P}_2$, the preconditioned matrices $\mathscr{A}\mathscr{P}_1^{-1}$ and $\mathscr{A}\mathscr{P}_2^{-1}$ have similar properties. For the preconditioned matrix $\mathscr{A}\mathscr{P}_1^{-1}$, the complex eigenvalues are contained in a circle with a radius which depends on ω. Choosing appropriate ω, only real eigenvalues occur. For the preconditioned matrix $\mathscr{A}\mathscr{P}_2^{-1}$, the eigenvalues are contained in a range which depends on ω, too. That is to say, appropriate ω can chosen such that the eigenvalues are more cluster and minimum residual method such as GMRES used to solve preconditioned linear system has better convergence rate than [22,16]. Our numerical experiments of a model Stokes problem are presented to show this.

Throughout this paper $\|\cdot\|$ indicates the 2-norm and $i = \sqrt{-1}$ denotes the imaginary unit. For a vector x, x^T and x^* indicate its transpose and transposed conjugate, respectively. For a matrix A, $A > 0$ means that A is symmetric and positive definite.

2. Eigenvalue analysis of preconditioned matrix $\mathcal{A}\mathcal{P}_1^{-1}$

We consider the eigenvalue problem

$$\mathscr{A}\mathscr{P}_1^{-1}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix} = \theta \begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix},$$

or equivalently, the generalized eigenvalue problem with $\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \mathscr{P}_1^{-1}\begin{bmatrix} \hat{u} \\ \hat{v} \end{bmatrix}$,

$$\mathscr{A}\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \theta \mathscr{P}_1 \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix}. \tag{2.1}$$

Consider additionally the block diagonal preconditioner

$$\mathscr{P}_0 = \begin{bmatrix} \widehat{A} & 0 \\ 0 & \widehat{C} \end{bmatrix}. \tag{2.2}$$

Denote $\begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \mathscr{P}_0^{-\frac{1}{2}}\begin{bmatrix} u \\ v \end{bmatrix}$ since $\mathscr{P}_0$ is symmetric and positive definite. Then we can write the generalized eigenvalue problem (2.1) as

$$\mathscr{P}_0^{-\frac{1}{2}}\mathscr{A}\mathscr{P}_0^{-\frac{1}{2}}\begin{bmatrix} u \\ v \end{bmatrix} = \theta \mathscr{P}_0^{-\frac{1}{2}}\mathscr{P}_1\mathscr{P}_0^{-\frac{1}{2}}\begin{bmatrix} u \\ v \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} \widehat{A}^{-\frac{1}{2}}A\widehat{A}^{-\frac{1}{2}} & \widehat{A}^{-\frac{1}{2}}B^T\widehat{C}^{-\frac{1}{2}} \\ \widehat{C}^{-\frac{1}{2}}B\widehat{A}^{-\frac{1}{2}} & -\widehat{C}^{-\frac{1}{2}}C\widehat{C}^{-\frac{1}{2}} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \theta \begin{bmatrix} I & \omega\widehat{A}^{-\frac{1}{2}}B^T\widehat{C}^{-\frac{1}{2}} \\ 0 & -I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.3}$$

Let $\widetilde{A} = \widehat{A}^{-\frac{1}{2}}A\widehat{A}^{-\frac{1}{2}}$, $\widetilde{B} = \widehat{C}^{-\frac{1}{2}}B\widehat{A}^{-\frac{1}{2}}$, $\widetilde{C} = \widehat{C}^{-\frac{1}{2}}C\widehat{C}^{-\frac{1}{2}}$. Then (2.3) becomes

$$\begin{bmatrix} \widetilde{A} & \widetilde{B}^T \\ \widetilde{B} & -\widetilde{C} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \theta \begin{bmatrix} I & \omega\widetilde{B}^T \\ 0 & -I \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}. \tag{2.4}$$