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# Global smoothness and uniform convergence of smooth Poisson–Cauchy type singular operators

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ABSTRACT

In this article we introduce the smooth Poisson–Cauchy type singular integral operators over the real line. Here we study their simultaneous global smoothness preservation property with respect to the $L_p$ norm, $1 \leqslant p \leqslant \infty$, by involving higher order moduli of smoothness. Also we study their simultaneous approximation to the unit operator with rates involving the modulus of continuity with respect to the uniform norm. The produced Jackson type inequalities are almost sharp containing elegant constants, and they reflect the high order of differentiability of the engaged function.



## 1. Introduction

The global smoothness preservation property of other singular integrals has been studied initially in [1] and later in [2]. The rate of convergence of these singular integrals has been studied initially in [3–5] and later in [6–8]. Most recently it was studied in detail in [9–11], over the real line, for the Picard general type integral operators and in [12,13] for the Gauss–Weierstrass type operators. All the above-mentioned papers along with the earlier ones [14–16] by the first author motivate the current work. Other motivation comes from [21,22]. Reference [18] is used for the basic calculations.

More precisely here we study the smooth Poisson–Cauchy singular integral operators over $\mathbb{R}$ acting on highly smooth functions. We study first their simultaneous global smoothness preservation property with respect to $\|\cdot\|_p$, $1 \leqslant p \leqslant \infty$, by using higher order moduli of smoothness. Then we study their simultaneous pointwise and uniform approximation to the unit operator with rates by using the first modulus of continuity. The established estimates are almost optimal and contain nice constants. The modulus of continuity in the estimates is with respect to the higher order derivative of the engaged function. The discussed operators are not in general positive.

## 2. Global smoothness preservation results

Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and consider the Lebesgue integral

$$M_\xi(f;x) := \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\frac{f(x+t)}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt, \tag{1}$$

$\xi > 0,\ \alpha \in \mathbb{N},\ \beta > \frac{1}{2\alpha},\ x \in \mathbb{R}.$

We notice, first, that we have

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$$M_\xi(c;x) = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\frac{c}{\left(t^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dt = c, \tag{2}$$

for any constant $c \in \mathbb{R}$.

We present the following result regarding global smoothness preservation properties of $M_\xi$.

**Theorem 1.** *Let $h > 0$.*

*(i) Assume that $\omega_m(f,h) < \infty$ and $M_\xi(f;x) \in \mathbb{R}$, then*

$$\omega_m(M_\xi f, h) \leqslant \omega_m(f,h). \tag{3}$$

*Inequality (3) is sharp, namely it is attained by $f(x) = x^m$.*

*(ii) Let $f \in L_1(\mathbb{R})$ then*

$$\omega_m(M_\xi f, h)_1 \leqslant \omega_m(f,h)_1. \tag{4}$$

*Finally,*

*(iii) let $f \in L_p(\mathbb{R}),\ p > 1$, then*

$$\omega_m(M_\xi f, h)_p \leqslant \omega_m(f,h)_p. \tag{5}$$

*Above we use for $m \in \mathbb{N}$ the mth modulus of smoothness for $1 \leqslant p \leqslant \infty$,*

$$\omega_m(f,h)_p := \sup_{0\leqslant t\leqslant h}\|\Delta_t^m f(x)\|_{p,x}, \tag{6}$$

*where*

$$\Delta_t^m f(x) := \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt), \tag{7}$$

*see also [17, p. 44]. Denote $\omega_m(f,h)_\infty = \omega_m(f,h)$.*

**Proof**

(i) We have the following:

$$\begin{aligned}
\Delta_t^m(M_\xi f(x)) &= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}M_\xi f(x+jt)\\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}\frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}f(x+jt+k)\frac{1}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk\\
&= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+jt+k)\right)\frac{1}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk\\
&= \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\Delta_t^m f(x+k)\frac{1}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk.
\end{aligned} \tag{8}$$

Therefore

$$|\Delta_t^m(M_\xi f(x))| \leqslant \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}|\Delta_t^m f(x+k)|\frac{1}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk. \tag{9}$$

From which we derive

$$\omega_m(M_\xi f,h) \leqslant \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\omega_m(f,h)\frac{1}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk = \omega_m(f,h), \tag{10}$$

proving (3).

To check the sharpness of (3) notice that $\omega_m(x^m,h)_\infty = m!h^m$. Also, by (8), we have that

$$\Delta_t^m(M_\xi f)(x) = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\frac{\Delta_t^m(x+k)^m}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk = \frac{\Gamma(\beta)\alpha\xi^{2\alpha\beta-1}}{\Gamma\left(\frac{1}{2\alpha}\right)\Gamma\left(\beta-\frac{1}{2\alpha}\right)}\int_{-\infty}^{\infty}\frac{m!t^m}{\left(k^{2\alpha}+\xi^{2\alpha}\right)^{\beta}}dk = m!t^m, \tag{11}$$

and therefore obtaining $\omega_m(M_\xi f,h) = m!h^m$.