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Existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay ☆

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ABSTRACT

This paper investigates the existence and uniqueness theorem of solutions to neutral stochastic differential equations with infinite delay (short for INSFDEs) at a space $BC((-\infty,0];R^d)$. Under the uniform Lipschitz condition, linear growth condition is weaken to obtain the moment estimate of the solution for INSFDEs. Furthermore, the existence, uniqueness theorem of the solution for INSFDEs is derived, and the estimate for the error between approximate solution and exact solution is given. On the other hand, under the linear growth condition, the uniform Lipschitz condition is replaced by the local Lipschitz condition, the existence, uniqueness theorem is also valid for INSFDEs on $[t_0,T]$. Moreover, the existence, uniqueness theorem still holds on interval $[t_0,\infty)$, where $t_0\in R$ is an arbitrary real number.

1. Introduction

Many dynamical systems not only depend on present and past states but also involve derivatives with delays. Hale [3] have studied deterministic neutral differential delay equations (NDDEs) and their stability. Taking the environmental disturbances into account, neutral stochastic differential equations was introduced by Kolmanovskii and Nosov [8]. They also investigated the stability and asymptotic stability of the equations (see also [11] or [12]). Mao has discussed neutral stochastic functional differential equations with finite delay (short for NSFDEs)

$$d[X(t)-D(X_t)]=f(X_t,t)dt+g(X_t,t)dB(t), \tag{1.1}$$

where $X_t=\{X(t+\theta):-\tau\leqslant\theta\leqslant 0\}$ could be considered as a $C([-\tau,0];R^d)$-valued stochastic process. The initial value of (1.1) was proposed as follows:

$$\begin{aligned}&X_{t_0}=\{\xi(\theta):-\tau\leqslant\theta\leqslant 0\}\text{is an }\mathscr{F}_{t_0}-\text{measurable}\\&C([-\tau,0];R^d)-\text{valued random variable such that }E\|\xi\|^2<\infty.\end{aligned} \tag{1.2}$$

For system (1.1), if uniform Lipschitz condition (1.3) and linear growth (1.4) are satisfied, that is, for any $\phi,\psi\in C([-\tau,0];R^d)$ and $t\in[t_0,T]$, it follows that

$$|f(\phi,t)-f(\psi,t)|^2\vee|g(\phi,t)-g(\psi,t)|^2\leqslant K\|\phi-\psi\|^2, K>0, \tag{1.3}$$

$$|f(\phi,t)|^2\vee|g(\phi,t)|^2\leqslant K(1+\|\phi\|^2), K>0; \tag{1.4}$$

then (1.1) had a unique solution $X(t)$, moreover $X(t)\in\mathscr{M}^2([t_0-\tau,T];R^d)$.

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Namely, the existence and uniqueness results for NSFDEs had been studied by [1], the related results of existence and uniqueness of solution could also be found in the literature [2,5,7]. At the same time, there are many literatures focus on stability for NSFDEs [4,6,8–10]. However, so far little is known about the existence and uniqueness results for neutral stochastic functional differential equations with infinite delay (short for INSFDEs). Motivated by the work [1] of Mao, we will derive the existence and uniqueness theorem of solution for INSFDEs at a phase space $BC([-\infty,0];R^d)$ in this paper. We still take $t_0 \in R$ as our initial time throughout this paper. Now, let us state our main results as follows: First, under the uniform Lipschitz condition and the linear growth condition is weaken to obtain the moment estimate of the solution for INSFDEs. Furthermore, the existence, uniqueness theorem of the solution for INSFDEs is derived, and the estimate for the error between approximate solution and exact solution is given. On the other hand, under the linear growth condition and the local lipschitz condition, the existence and uniqueness theorem is also valid for INSFDEs on the closed interval $[t_0,T]$. Moreover, the existence, uniqueness theorem still holds on the entire interval $[t_0,\infty)$.

2. Preliminary

In this paper, we adopt the symbols as follow: R^d denotes the usual d-dimensional Euclidean space, $|\cdot|$ denotes norm in R^d. If A is a vector or a matrix, its transpose is denoted by A^T; if A is a matrix, its trace norm is represented by $|A| = \sqrt{trace(A^TA)}$. Let $(\Omega,\mathscr{F},P)$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\in[t_0,+\infty)}$ satisfying the usual conditions (i.e. it is increasing and right continuous). $\mathscr{F}_{t_0}$ contains all P-null sets. $B(t)$ is a given m-dimensional standard Brownian motion. $\mathscr{F}_{t_0}$ is independent of the σ-field generated by $\{B(t)-B(t_0): t_0 \leqslant t \leqslant T\}$. Let $BC((-\infty,0];R^d)$ denote the family of bounded continuous R^d-value functions ϕ defined on $(-\infty,0]$ with the norm $\|\Phi\| = \sup_{-\infty<\theta\leqslant 0}|\Phi(\theta)|$.

Consider a d-dimensional neutral stochastic functional differential equations

$$d[X(t)-D(X_t)] = f(X_t,t)dt + g(X_t,t)dB(t), \tag{2.1}$$

where $X_t = \{X(t+\theta): -\infty<\theta\leqslant 0\}$ can be regarded as a $BC((-\infty,0];R^d)$-valued stochastic process, where $D: BC((-\infty,0];R^d)\to R^d$, $f: BC((-\infty,0];R^d)\times[t_0,T]\to R^d$ and $g: BC((-\infty,0];R^d)\times[t_0,T]\to R^{d\times m}$. The initial value is followed:

$$\begin{aligned}&X_{t_0} = \xi = \{\xi(\theta): -\infty<\theta\leqslant 0\} \text{ is } \mathscr{F}_{t_0}-\text{measurable}\\ &BC((-\infty,0];R^d)-\text{valued random variable such that } \xi\in\mathscr{M}^2((-\infty,0];R^d)\end{aligned} \tag{2.2}$$

where $\mathscr{M}^2((-\infty,0];R^d)$ denotes the family of the process $\{\xi(t)\}_{t\leqslant 0}$ in $L^p((-\infty,0];R^d)$ such that $E\int_{-\infty}^0|\xi(t)|^2dt<\infty$ a.s. Our purpose is to find the solution of (2.1) with initial value (2.2). Hence, we will show the definition of the solution of (2.1), the existence and uniqueness theorem and the estimate for approximate solution in the next section.

3. The existence and uniqueness theorem

Lemma 3.1. *If $p>2, g\in\mathscr{M}^2([t_0,T];R^{d\times m})$ such that $E\int_{t_0}^T|g(s)|^p ds<\infty$, then*

$$E\left|\int_{t_0}^T g(s)dB(s)\right|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{p/2} T^{\frac{p-2}{2}} E\int_{t_0}^T |g(s)|^p ds.$$

The proof of Lemma 3.1 can be found in [1].

Definition 3.1. R^d-value stochastic process $X(t)$ defined on $-\infty<t\leqslant T$ is called a solution of (2.1) with initial value (2.2), if $X(t)$ has the following properties:

(i) $X(t)$ is continuous and $\{X(t)\}_{t_0\leqslant t\leqslant T}$ is $\mathscr{F}_t$-adapted;
(ii) $\{f(X_t,t)\}\in\mathscr{L}^1([t_0,T],R^d)$ and $\{g(X_t,t)\}\in\mathscr{L}^2([t_0,T];R^{d\times m})$;
(iii) $X_{t_0}=\xi$ for each $t_0\leqslant t\leqslant T$, $X(t) = \xi(0) + D(X_t) - D(X_{t_0}) + \int_{t_0}^t f(X_s,s)ds + \int_{t_0}^t g(X_s,s)dB(s)$ a.s.

$X(t)$ is named a unique solution, if any other solution $\widetilde{X}(t)$ is distinguishable with $X(t)$, that is

$$P\{X(t)=\widetilde{X}(t),\ \text{for any} -\infty<t\leqslant T\} = 1.$$

Let us now begin to establish the theory of the existence and uniqueness for (2.1) with initial value (2.2). Under uniform Lipschitz condition, linear growth condition is weaken then follows Theorem 3.1.

Theorem 3.1. *Assume that there exist two positive constants $\overline{K}$ and K such that*

(i) (uniform Lipschitz condition) *For all $\phi,\psi\in BC((-\infty,0];R^d)$ and $t\in[t_0,T]$,*

$$|f(\phi,t)-f(\psi,t)|^2 \vee |g(\phi,t)-g(\psi,t)|^2 \leqslant \overline{K}\|\phi-\psi\|^2, \tag{3.1}$$