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Boundary value problems for differential equations with deviated arguments which depend on the unknown solution

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ABSTRACT

We discuss boundary value problems for first-order functional differential equations with deviated arguments which depend on the unknown solution. We formulate sufficient conditions for existence of a quasisolution and a unique solution of such problems. To obtain the results we use the method of monotone iterations.

1. Introduction

Let us consider the problem

$$\begin{cases} x'(t) = f\left(t, x(\beta(t,x(t))), \int_0^t g(t,s,x(s))ds\right) \equiv F(x,x,x)(t), & t \in J, \\ x(0) = \lambda x(T) + k, \end{cases} \tag{1}$$

where $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \beta \in C(J \times \mathbb{R}, \mathbb{R}), g \in C(J \times J \times \mathbb{R}, \mathbb{R}), \lambda, k \in \mathbb{R}, J = [0,T]$ and

$$F(x,y,z)(t) = f\left(t, x(\beta(t,y(t))), \int_0^t g(t,s,z(s))ds\right).$$

In this paper we extend some results of paper [3] where function f did not depend on the third variable. Note that the deviating argument β depends on the unknown solution x.

The plan of this paper is as follows: In Section 2, we formulate conditions which guarantee existence of maximal and minimal quasisolution of problem (1) in a corresponding sector. To prove the existence results we apply the monotone iterative method; for details see for example [4]. See also [1,2,5]. In Section 3, we formulate sufficient conditions under which problem (1) has a unique solution. In the last section we give an example to illustrate the applications of obtained results.

2. Quasisolution of problem (1)

In this section we investigate problem (1) when it has a quasisolution. We consider two cases.

2.1. Case 1: $\lambda \geqslant 0$

A pair $u, v \in C^1(J, \mathbb{R})$ is called a quasisolution of problem (1) if

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$$\begin{cases} u'(t) = F(v,v,v)(t), \ t \in J, & u(0) = \lambda u(T) + k, \\ v'(t) = F(u,u,u)(t), \ t \in J, & v(0) = \lambda v(T) + k. \end{cases}$$

A pair $U, V \in C^1(J,\mathbb{R})$ is called the minimal and maximal quasisolution of problem (1) if for any $u, v \in C^1(J,\mathbb{R})$ quasisolution of (1) we have $U(t) \leqslant u(t), v(t) \leqslant V(t), t \in J$.

Theorem 1. *Assume that:*

1. $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \beta \in C(J \times \mathbb{R}, \mathbb{R}), g \in C(J \times J \times \mathbb{R}, \mathbb{R})$ and f is nonincreasing with respect to the last two variables
2. a pair $y_0, z_0 \in C^1(J,\mathbb{R})$ satisfies the system:

$$\begin{cases} y_0'(t) \leqslant F(z_0,z_0,z_0)(t), \ t \in J, & y_0(0) \leqslant \lambda y_0(T) + k, \\ z_0'(t) \geqslant F(y_0,y_0,y_0)(t), \ t \in J, & z_0(0) \geqslant \lambda z_0(T) + k, \end{cases} \tag{2}$$

3. $y_0(t) \leqslant z_0(t), t \in J$
4. $\beta : \Omega \to J, g : J \times \Omega \to \mathbb{R}$ where $\Omega = \{(t,u) : y_0(t) \leqslant u \leqslant z_0(t), \ t \in J\}$, are nondecreasing with respect to u for $y_0(t) \leqslant u \leqslant z_0(t), t \in J$,
5. y_0, z_0 are nondecreasing on J and $f(t,u,v) \geqslant 0$ for $t \in J, y_0(t) \leqslant u \leqslant z_0(t), \int_0^t g(t,s,y_0(s))\,ds \leqslant v \leqslant \int_0^t g(t,s,z_0(s))\,ds, \ t \in J$.

Then, in the sector $[y_0,z_0]_* = \left\{u \in C^1(J,\mathbb{R}) : y_0(t) \leqslant u(t) \leqslant z_0(t), t \in J\right\}$, *problem (1) has the minimal and maximal quasisolution.*

Proof. Let us define sequences $\{y_n, z_n\}$ by

$$\begin{cases} y_{n+1}'(t) = F(z_n,z_n,z_n)(t), & t \in J, \quad y_{n+1}(0) = \lambda y_n(T) + k, \\ z_{n+1}'(t) = F(y_n,y_n,y_n)(t), & t \in J, \quad z_{n+1}(0) = \lambda z_n(T) + k \end{cases}$$

for $n = 0, 1, \ldots$ Note that in view of assumption 5 functions $y_n, z_n, n \in \mathbb{N}$, are nondecreasing on J.

First we show that

$$y_0(t) \leqslant y_1(t) \leqslant z_1(t) \leqslant z_0(t), \quad t \in J. \tag{3}$$

Put $p = y_0 - y_1$. In view of (2) we have $p(0) \leqslant 0$ and $p'(t) \leqslant 0$. Hence $p(t) \leqslant 0, t \in J$ and $y_0(t) \leqslant y_1(t)$ on J. Analogically we can show that $z_1(t) \leqslant z_0(t)$.

Now put $p = y_1 - z_1$. We have

$$p(0) = \lambda[y_0(T) - z_0(T)] \leqslant 0$$

and

$$p'(t) = F(z_0,z_0,z_0)(t) - F(y_0,y_0,y_0)(t) \leqslant 0$$

because

$$y_0(\beta(t,y_0(t))) \leqslant z_0(\beta(t,z_0(t)))$$

and

$$\int_0^t g(t,s,y_0(s))\,ds \leqslant \int_0^t g(t,s,z_0(s))\,ds$$

in view of assumptions 4 and 5. It yields that $p(t) \leqslant 0$ on J and relation (3) holds.

Note that

$$y_1'(t) = F(z_0,z_0,z_0)(t) - F(z_1,z_1,z_1)(t) + F(z_1,z_1,z_1)(t) \leqslant F(z_1,z_1,z_1)(t)$$

$$z_1'(t) = F(y_0,y_0,y_0)(t) - F(y_1,y_1,y_1)(t) + F(y_1,y_1,y_1)(t) \geqslant F(y_1,y_1,y_1)(t)$$

because

$$z_0(\beta(t,z_0(t))) \geqslant z_1(\beta(t,z_1(t))), \quad \int_0^t g(t,s,z_0(s))\,ds \geqslant \int_0^t g(t,s,z_1(s))\,ds$$

and

$$y_0(\beta(t,y_0(t))) \leqslant y_1(\beta(t,y_1(t))), \quad \int_0^t g(t,s,y_0(s))\,ds \leqslant \int_0^t g(t,s,y_1(s))\,ds.$$

Moreover

$$y_1(0) \leqslant \lambda y_1(T) + k \quad \text{and} \quad z_1(0) \geqslant \lambda z_1(T) + k.$$

Thus y_1, z_1 satisfy system (2).