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# Exact solutions of a class of second-order nonlocal boundary value problems and applications

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ABSTRACT

In this paper, solutions of a class of second-order differential equations with some multi-point boundary conditions are studied. We give exact expressions of the solutions for the linear $m$-point boundary problems by the Green's functions. As applications, we study uniqueness and iteration of the positive solutions for a nonlinear singular second-order $m$-point boundary value problem.



## 1. Introduction

The methods using Green's functions can solve some mechanics problems – oscillation and wave power, for example. The exact expressions of the solutions for some linear ordinary differential equations boundary value problems can be denoted by Green's functions of the problems. Some boundary value problems for nonlinear differential equations can be transformed into the nonlinear integral equations the kernel of which are the Green's functions of corresponding linear differential equations. The integral equations can be solved by to investigate the property of the Green's functions. The concept, the significance and the development of Green's functions can be seen in [1].

The theory of multi-point boundary value problems has been emerging as an important area of investigations in recent years. The investigation on multi-point boundary value problems with the equation $u''(t) + f(t,u) = 0$ can be seen in [2–9] and its references. The investigation on $m$-point boundary value problems with one-dimensional $p$-Laplacian can be seen in [10–16] and its references.

In this paper, we consider the second-order linear differential equation

$$-u'' + k^2 u = f(t), \quad t \in [a,b] \tag{1.1}$$

subject to the boundary value conditions, respectively,

(i) $u(a) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u'(b) = 0$;
(ii) $u(a) = 0, \quad u'(b) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$;
(iii) $u(a) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i), \quad u'(b) = 0$;
(iv) $u(a) = 0, \quad u'(b) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i)$;
(v) $u'(a) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), \quad u(b) = 0$;
(vi) $u'(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i)$;
(vii) $u'(a) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i), \quad u(b) = 0$;
(viii) $u'(a) = 0, \quad u(b) = \sum_{i=1}^{m-2} \alpha_i u'(\eta_i)$,

where $k \neq 0,\ m > 2,\ a < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < b,\ \alpha_i\ (i = 1, 2, \ldots, m-2)$ are given numbers.

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This paper is organized as follows. In Section 2, we study solutions of the Eq. (1.1) satisfying the boundary conditions (i) and give the expression of the unique solution by the Green's function, that incarnate the general method of deriving the Green's functions for a class of boundary problems. In Section 3, for some interrelated boundary conditions, we give solutions of the linear boundary value problems directly, omitting the particular of derivation. The correctness of the solutions only need direct verification. As applications, in Section 4 we study uniqueness of the solutions, the iteration and the rate of convergence by the iteration for a nonlinear singular second-order $m$-point boundary value problem.

Throughout the paper, we denote the hyperbolic sine function and the hyperbolic cosine function by $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$, respectively.

## 2. Main results and proof

**Theorem 2.1.** *Suppose the function $f(t)$ is continuous on $[a,b]$ and in addition assume $k \neq 0$, $\cosh(k(b-a)) \neq \sum_{i=1}^{m-2} \alpha_i \cosh(k(b-\eta_i))$, then the second-order m-point linear boundary value problem*

$$\begin{cases} -u''(t) + k^2 u(t) = f(t), & a \leqslant t \leqslant b, \\ u(a) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), & u'(b) = 0 \end{cases} \tag{2.1}$$

*has a unique solution*

$$u(t) = \int_a^b F_1(t,s) f(s) ds,$$

*where*

$$F_1(t,s) = G(t,s) + \frac{\cosh(k(b-t))}{\cosh(k(b-a)) - \sum_{i=1}^{m-2} \alpha_i \cosh(k(b-\eta_i))} \sum_{i=1}^{m-2} \alpha_i G(\eta_i, s) \tag{2.2}$$

*with*

$$G(t,s) = \begin{cases} \dfrac{\sinh(k(s-a)) \cosh(k(b-t))}{k \cosh(k(b-a))}, & \text{if } a \leqslant s \leqslant t, \\ \dfrac{\sinh(k(t-a)) \cosh(k(b-s))}{k \cosh(k(b-a))}, & \text{if } t \leqslant s \leqslant b. \end{cases} \tag{2.3}$$

**Remark 1.** We call $F_1(t,s)$ Green's function of the boundary value problem (2.1).

**Proof.** It is well known that the Green's function is $G(t,s)$ as in Eq. (2.3) for the second-order two-point linear boundary value problem

$$\begin{cases} -w''(t) + k^2 w(t) = f(t), & t \in [a,b], \\ w(a) = 0, & w'(b) = 0, \end{cases} \tag{2.4}$$

and the solution of (2.4) is given by

$$w(t) = \int_a^b G(t,s) f(s) ds \tag{2.5}$$

and

$$w(a) = 0, \quad w'(b) = 0, \quad w(\eta_i) = \int_a^b G(\eta_i, s) f(s) ds. \tag{2.6}$$

Assume that $u(t)$ is a solution of (2.1), and let

$$z(t) = u(t) - w(t), \quad t \in [a,b] \tag{2.7}$$

with $w(t)$ as in (2.5). Then

$$\begin{cases} -u''(t) + k^2 u(t) = f(t), \\ u(a) = \sum_{i=1}^{m-2} \alpha_i u(\eta_i), & u'(b) = 0, \end{cases} \tag{2.8}$$

$$-z''(t) + k^2 z(t) = 0.$$

Therefore

$$z(t) = \alpha e^{kt} + \beta e^{-kt}, \tag{2.9}$$